Weiss–Weinstein Lower Bounds for Markovian Systems. Part 2: Applications to Fault-Tolerant Filtering

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***Abstract*—Characterized by sudden structural changes, fault-prone systems are modeled using the framework of systems with switching parameters or hybrid systems. Since a closed-form mean-square optimal filtering algorithm for this class of systems does not exist, it is of particular interest to derive a lower bound on the state estimation error covariance. The well known Cramér–Rao bound is not applicable to fault-prone systems because of the discrete distribution of the fault indicators, which violates the regularity conditions associated with this bound. On the other hand, the Weiss–Weinstein lower bound is essentially free from regularity conditions. Moreover, a sequential version of the Weiss–Weinstein bound, suitable for Markovian dynamic systems, is presented by the authors in a companion paper. In the present paper, this sequential version is applied to several classes of fault-prone dynamic systems. The resulting bounds can be used to examine fault detectability and identifiability in these systems. Moreover, it is shown that several recently reported lower bounds for fault-prone systems are special cases of, or closely related to, the sequential version of the Weiss–Weinstein lower bound.**

***Index Terms*—Estimation error lower bound, fault detection and isolation, hybrid systems.**

I. Introduction

Modern multisensor applications, such as navigation and target tracking systems, require the fusion of data acquired by a large number of different sensors. In many situations, the sensors might be subjected to faults, either due to internal malfunctions, or because of external interferences. These sensor faults are usually manifested as a sudden addition of noise (white or colored) to the sensor measurements, or even interruption of the output signal. Thus, global positioning system (GPS) jamming and spoofing signals take, in general, the form of colored noises and appear suddenly whenever they are activated [1]. Magnetometer faults, caused by magnetic fields generated by spacecraft electronics and electromagnetic torquing coils, usually take the form of biases [2, p.251] and appear whenever the corresponding current starts. Rate gyro faults, caused by input accelerations if the gyro gimbals are not perfectly balanced, are also usually modeled as biases [2, p.198] and appear whenever the platform accelerates. In view of present day systems' high accuracy requirements, the problem of fault-tolerant filtering in multisensor systems is of major importance.

Characterized by sudden structural changes, fault-prone systems are usually modeled and analyzed using the framework of hybrid systems [3, p. 177]. The total state of these systems comprises two kinds of parameters: the continuously distributed parameters, usually referred to as the states of the system, and a Markovian switching parameter, which takes values in a finite set and is referred to, in general, as the system mode. Considering fault-prone systems, one of the switching parameter values corresponds to the nominal system operation, whereas the others represent various fault conditions [1], [4]. In systems with independent fault sources, such as GPS-aided inertial navigation systems, the aforementioned model can be simplified: the faults caused by different sources can be modeled as separate Markovian Bernoulli random processes, where "1" stands for a fault situation and "0" stands for no fault situation.

It is well known that the mean-square optimal filtering algorithm for hybrid systems, which provides the estimates of the state vector and the switching parameters, requires infinite computation resources [5]. Therefore, a variety of suboptimal estimation techniques was proposed [6]–[11]. Since the estimates of the state vector and the fault indicators are suboptimal, it is of particular interest to obtain some measure of their efficiency. The natural means for this purpose is the comparison to a lower bound on the estimation error.

The most popular lower bound is the well known Cramér–Rao lower bound (CRLB). This bound was presented in [12, p. 84] in the context of Bayesian estimation of static random parameters. Recently, a sequential version of the CRLB has been derived [13], which makes it suitable for general Markovian dynamic systems. Together with the static form of the CRLB, these results served as a basis for a large number of applications [14]–[18].

Unfortunately, albeit being a very useful tool for systems with continuously distributed parameters, the CRLB cannot be directly calculated for both the state and the mode variables of a hybrid system. The reason lies in the fact that the system of interest must satisfy the CRLB regularity conditions, which, even in their weakest form, state that all associated probability density functions (pdf's) must be continuously differentiable [19]. Clearly, this is not the case in hybrid systems in general, and in fault-prone systems in particular, since the mode variables, or fault indicators, are discrete Markovian sequences. The CRLB can be applied directly only to the state vector (see, e.g., [18]),

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which is continuously distributed and, in most cases, satisfies the regularity conditions. However, the application of the sequential form of the CRLB [13] is based on a special requirement regarding the structure of the system measurements, which is not satisfied by a general class of fault-prone systems. Another approach is based on treating the discretely distributed fault indicators as nuisance parameters, known to the observer. Originally proposed in [20] for a general class of systems, this approach was applied in [21] to target tracking using sensors with detection probability smaller than one. Using the fact that the measurement interruption process is white, the authors derived an approximation of the CRLB for the tracking errors. However, in the case of general (nonwhite) Markov sequences, such derivation becomes cumbersome. Moreover, this lower bound cannot be evaluated in closed form, calling for the use of extensive Monte-Carlo simulations.

An indirect application of the sequential CRLB [13] has been recently presented for systems with fault-prone measurements [22]. The lower bound was established through approximating the discrete distribution of the fault indicators using a continuous one, then using a limiting procedure. The results facilitate a relatively simple calculation of a lower bound for the continuously distributed states of the system; however, the associated lower bound for the discretely distributed fault indicators is trivially zero. An alternative, nontrivial lower bound for the fault indicators in systems with fault-prone measurements has been recently presented in [23].

There is still another, more powerful bound, known in the literature as the Weiss–Weinstein lower bound (WWLB) [19]. Both the CRLB and the Bobrovsky–Zakai lower bound [24] are special cases of the WWLB [25]. Being essentially free from regularity conditions, the WWLB can be applied to a very large class of estimation problems, including estimation of discretely distributed parameters. Moreover, based upon an earlier conference paper [26], the companion paper by the authors [27] presents a sequential version of the WWLB. Making use of the transitional state distribution, the measurement conditional distribution, and the marginal state distribution renders the resulting class of WWLBs practical for Markovian dynamic systems.

This paper presents applications of the sequential version of the WWLB [27] to several classes of fault-prone systems. It is shown that the lower bound presented in [22], and some results obtained in [23], are special cases of the new sequential lower bound. Moreover, since the bound is suitable for a very general fault architecture, it can be used to examine fault detectability and identifiability in complex fault-prone systems.

The remainder of this paper is organized as follows. The sequential version of the WWLB, presented by the authors in the companion paper [27], is reviewed in Section II. The new lower bound is applied to a general class of fault-prone systems in Section III. Then, several special cases are considered. A white fault sequence case is discussed in Section IV. Systems whose state process consists of a single Bernoulli Markov chain are addressed in Section V. In Section VI, the new bound is applied to systems where the continuously distributed states are insensitive to faults. The result is related to the CRLB-type lower bound recently presented in [22]. Finally, Section VII addresses systems with fault-free main dynamics and independent fault-prone measurement channels. The corresponding result is related to the lower bound for fault indicators recently presented in [23]. Concluding remarks are offered in the last section. Some auxiliary calculations and developments, which are detailed in [28], are omitted for conciseness.

II. Review of the Sequential WWLB

Consider a dynamic system, characterized by a Markovian state process $\{z_k\}_{k=0}^{\infty}$, which is measured through a measurement process $\{y_k\}_{k=1}^{\infty}$, where $z_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^m$. The joint distribution of the time histories of z_k and y_k is given by

$$p_{\mathcal{Z}_k,\mathcal{Y}_k}(\Xi_k, \Upsilon_k) = \prod_{l=0}^{k} p_{y_l|z_l}(Y_l \mid Z_l) p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) \quad (1)$$

where $\mathcal{Z}_k \triangleq \{z_0, z_1, \ldots, z_k\}$ and $\mathcal{Y}_k \triangleq \{y_0, y_1, \ldots, y_k\}$ are the state and measurement histories, respectively, $\Xi_k \triangleq \{Z_0, Z_1, \ldots, Z_k\}$ and $\Upsilon_k \triangleq \{Y_0, Y_1, \ldots, Y_k\}$ are their realizations, respectively, and

$$p_{z_0|z_{-1}}(Z_0 \mid Z_{-1}) = p_{z_0}(Z_0) \quad (2a)$$

$$p_{y_0|z_0}(Y_0 \mid Z_0) = p_{y_0}(Y_0). \quad (2b)$$

Define

$$L_l\left(Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right) \triangleq \frac{p_{y_l|z_l}\left(Y_l \mid Z_l^{(1)}\right) p_{z_l|z_{l-1}}\left(Z_l^{(1)} \mid Z_{l-1}\right)}{p_{y_l|z_l}\left(Y_l \mid Z_l^{(2)}\right) p_{z_l|z_{l-1}}\left(Z_l^{(2)} \mid Z_{l-1}\right)} \quad (3a)$$

$$K_l\left(Z_{l+1}, Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right) \triangleq \frac{p_{z_{l+1}|z_l}\left(Z_{l+1} \mid Z_l^{(1)}\right)}{p_{z_{l+1}|z_l}\left(Z_{l+1} \mid Z_l^{(2)}\right)} L_l\left(Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right). \quad (3b)$$

Finally, for some time instant l and some vector $h(l) \in \mathbb{R}^n$, define

$$\mathcal{A}_{h(l)} \triangleq \left\{Z_{l-1} \text{ such that } \{Z_{l-1} \in \operatorname{supp} p_{z_{l-1}}\} \wedge \{Z_{l-1} + h(l) \in \operatorname{supp} p_{z_{l-1}}\}\right\} \quad (4)$$

where $\wedge$ denotes the logical conjunction operator, and $\operatorname{supp} p$ denotes the support of p, as in equation (5), shown at the bottom of the page. Then, the sequential WWLB bound, presented in

$$\alpha_l(h(l), Z_{l-1}) \triangleq \int_{-\infty}^{+\infty} \sqrt{p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1} + h(l)) p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1})} dZ_l. \quad (5)$$

the companion paper [27], applies to systems satisfying the following fundamental assumption.

Assumption 2.1: For every time instant l there exists $h(l) \in \mathbb{R}^n$ satisfying

$$\mathcal{A}_{h(l)} \neq \varnothing \tag{6}$$

such that

$$\alpha_l(h(l), Z_{l-1}) = \alpha_l(h(l)) \quad \forall\, Z_{l-1} \in \mathcal{A}_{h(l)} \tag{7}$$

Remark 2.1: The quantity $\alpha_l(h(l), Z_{l-1})$ is related to the sensitivity of the state transitional distribution $p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1})$ to changes in the value of the conditioning state vector z_{l-1}. Assumption 2.1 states that this sensitivity is uniform over all values of z_{l-1}.

In the companion paper [27], it is shown that Assumption 2.1 is satisfied for a large class of systems including systems with linear dynamics, Bernoulli Markov chains, and systems satisfying the CRLB regularity conditions.

The sequential version of the WWLB is given in the following Theorem.

Theorem 2.1: Consider a Markovian dynamic system. Let

$$H(l,l) \triangleq \left[h_l^{(1)} \quad h_l^{(2)} \quad \dots \quad h_l^{(n)} \right], \qquad l = 1, 2, \dots, k+1 \tag{8}$$

be a set of matrices, composed from such columns $h_l^{(i)} \in \mathbb{R}^n$, for which Assumption 2.1 holds. Then, the corresponding WWLB is given by

$$E\left[(z_{k+1} - \hat{z}_{k+1|k+1})(z_{k+1} - \hat{z}_{k+1|k+1})^T\right] \geqslant H(k+1,k+1) J_{k+1}^{-1} H(k+1,k+1)^T \tag{9}$$

where the matrix J_{k+1} is computed sequentially as

$$\begin{aligned} J_{k+1} = {} & G^{(k+1)}(k+1,k+1) \\ & - G^{(k+1)}(k+1,k) \\ & \times \left[J_k + G^{(k+1)}(k,k) - G^{(k)}(k,k) \right]^{-1} \\ & \times G^{(k+1)}(k,k+1) \end{aligned} \tag{10a}$$

$$J_0 = G^{(0)}(0,0) \tag{10b}$$

and the (i,j) entries of the matrices $G^{(k+1)}(\xi,\eta)$ are computed using equation (11), shown at the bottom of the next page.

III. General Fault-Prone System

Suppose that the state vector of a dynamic system can be partitioned as follows:

$$z_k = \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix}, \quad x_k \in \mathbb{R}^{n_x} \tag{12}$$

such that the following assumptions hold.

Assumption 3.1: The joint distribution of the state and the measurement histories can be factored as

$$p_{\mathcal{Z}_k,\mathcal{Y}_k}(\Xi_k, \Upsilon_k) = \prod_{l=0}^{k} p_{y_l|x_l,\gamma_l}(Y_l \mid X_l, \Gamma_l) \times p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1}, \Gamma_l)\, p_{\gamma_l|\gamma_{l-1}}(\Gamma_l \mid \Gamma_{l-1}) \tag{13}$$

where the pdf's $p_{y_l|x_l,\gamma_l}(Y_l \mid X_l, \Gamma_l)$ and $p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1}, \Gamma_l)$ satisfy the CRLB regularity conditions. In addition, x_0 and γ_0 are independent.

Assumption 3.2: The vector γ_k is assumed to have the form

$$\gamma_k = \left[\gamma_k^{(1)}, \gamma_k^{(2)}, \dots, \gamma_k^{(N)}\right]^T \tag{14}$$

where $\left\{\gamma_k^{(i)}\right\}_{k=0}^{\infty}$ are independent Bernoulli Markov chains with transition probabilities

$$P_{10}^{(i)}(l \mid l-1) = \Pr\left\{\gamma_l^{(i)} = 1 \mid \gamma_{l-1}^{(i)} = 0\right\} \tag{15a}$$

$$P_{11}^{(i)}(l \mid l-1) = \Pr\left\{\gamma_l^{(i)} = 1 \mid \gamma_{l-1}^{(i)} = 1\right\}. \tag{15b}$$

In other words, the transition pdf's of the elements of γ_k are

$$p_{\gamma_l^{(i)}|\gamma_{l-1}^{(i)}}\left(\Gamma_l^{(i)} \mid \Gamma_{l-1}^{(i)}\right) = P_{1,\Gamma_{l-1}^{(i)}}\, \delta\left(\Gamma_l^{(i)} - 1\right) + \left(1 - P_{1,\Gamma_{l-1}^{(i)}}\right) \delta\left(\Gamma_l^{(i)}\right) \tag{16}$$

where $\delta(\cdot)$ is Dirac's delta.

Assumption 3.3: The system has a symmetry property, in the sense that

$$p_{x_0}(-X_0) = p_{x_0}(X_0) \tag{17a}$$

$$p_{x_l|x_{l-1},\gamma_l}(-X_l \mid -X_{l-1}, \Gamma_l) = p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1}, \Gamma_l) \tag{17b}$$

$$p_{y_l|x_l,\gamma_l}(-Y_l \mid -X_l, \Gamma_l) = p_{y_l|x_l,\gamma_l}(Y_l \mid X_l, \Gamma_l). \tag{17c}$$

Assumption 3.3 is satisfied, e.g., in linear hybrid systems. Several properties of the symmetry assumption are summarized and proved in [28].

The model (12)–(16) can be used to describe systems with independent faults. In such systems, x_k is the state vector, and $\{\gamma_k^{(i)}\}_{i=1}^{N}$ are fault indicators. More specific examples of such systems are presented in Sections VI-A and VII-B. The interested reader is also referred to [22] and [23] for lower bounds derived for special cases of the system defined in (12)–(16).

To facilitate the calculation of the sequential WWLB in this case, the columns $h_l^{(\eta)}$ of the matrices $H(l,l)$ are selected as

$$h_l^{(\eta)} = \begin{cases} \left[\underbrace{0,\dots,0}_{\eta-1}, \varepsilon, \underbrace{0,\dots,0}_{n-\eta}\right]^T, \ \varepsilon \to 0, & \text{for } \eta \leqslant n_x \\ \left[\underbrace{0,\dots,0}_{\eta-1}, 1, \underbrace{0,\dots,0}_{n-\eta}\right]^T, & \text{for } \eta > n_x. \end{cases} \tag{18}$$

In other words, the matrices $H(l,l)$ are selected to be diagonal, with diagonal entries of $\varepsilon \to 0$ and 1, corresponding to the elements of x and γ, respectively. For convenience, the following notation will be used in the sequel:

$$\tilde{h}^{(i)} \triangleq \Big[\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{N-i}\Big]^T. \tag{19}$$

Remark 3.1: The selection of $H(l,l)$, given by (18), is not unique. Therefore, other lower bounds can be obtained for other selections.

Before applying Theorem 2.1, the validity of Assumption 2.1 needs to be verified. This is done in the following Theorem.

Theorem 3.1: Suppose a fault-prone dynamic system satisfies Assumptions 3.1 and 3.2. Then, Assumption 2.1 holds, with $h_l^{(\eta)}$ as in (18).

Proof: First, for $h_l^{(\eta)}$, $\eta \leqslant n_x$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \sqrt{p_{z_l|z_{l-1}}\left(Z_l \mid Z_{l-1}+h_{l-1}^{(\eta)}\right) p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1})}dZ_l \\
&= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\left(p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1},\Gamma_l) + \frac{1}{2}\frac{\partial p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1},\Gamma_l)}{\partial X_{l-1}^{(\eta)}}\varepsilon\right)dX_l + o(\varepsilon)\right] \\
&\quad\times p_{\gamma_l|\gamma_{l-1}}(\Gamma_l \mid \Gamma_{l-1})d\Gamma_l \\
&= \int_{-\infty}^{+\infty}(1+o(\varepsilon))\, p_{\gamma_l|\gamma_{l-1}}(\Gamma_l \mid \Gamma_{l-1})d\Gamma_l = 1+o(\varepsilon)
\end{aligned} \tag{20}$$

which does not depend on X_{l-1} and Γ_{l-1}. [Similarly to (14), $X_{l-1}^{(\eta)}$ in (20) denotes the ηth element of the vector X_{l-1}.] Now,

$$\begin{aligned}
&G^{(k+1)}(k+1,k+1)_{ij} \\
&= E\Bigg[\left(\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(i)},z_{k+1};z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1};z_{k+1}-h_{k+1}^{(i)},z_{k+1};z_k\right)}\right) \\
&\quad\times\left(\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(j)},z_{k+1};z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1};z_{k+1}-h_{k+1}^{(j)},z_{k+1};z_k\right)}\right)\Bigg] \\
&\quad\times\frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(i)},z_{k+1};z_k\right)}\right]E\left[\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(j)},z_{k+1};z_k\right)}\right]}
\end{aligned} \tag{11a}$$

$$\begin{aligned}
&G^{(k+1)}(k+1,k)_{ij} = G^{(k+1)}(k,k+1)_{ij}^T \\
&= E\Bigg[\left(\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(i)},z_{k+1};z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1};z_{k+1}-h_{k+1}^{(i)},z_{k+1};z_k\right)}\right) \\
&\quad\times\left(\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(j)},z_k;z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1},y_k;z_k-h_k^{(j)},z_k;z_{k-1}\right)}\right)\Bigg] \\
&\quad\times\frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1};z_{k+1}+h_{k+1}^{(i)},z_{k+1};z_k\right)}\right]E\left[\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(j)},z_k;z_{k-1}\right)}\right]}
\end{aligned} \tag{11b}$$

$$\begin{aligned}
&G^{(k+1)}(k,k)_{ij} \\
&= E\Bigg[\left(\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(i)},z_k;z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1},y_k;z_k-h_k^{(i)},z_k;z_{k-1}\right)}\right) \\
&\quad\times\left(\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(j)},z_k;z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1},y_k;z_k-h_k^{(j)},z_k;z_{k-1}\right)}\right)\Bigg] \\
&\quad\times\frac{1}{\left\{E\left[\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(i)},z_k;z_{k-1}\right)}\right]E\left[\sqrt{K_k\left(z_{k+1},y_k;z_k+h_k^{(j)},z_k;z_{k-1}\right)}\right]\right\}}
\end{aligned} \tag{11c}$$

refer to equation (21), as shown at the bottom of the page. The last expression is neither a function of X_{l-1} nor of $\gamma_{l-1}^{(j)}$, $j \neq i$. Moreover, following the arguments presented in [27] for Bernoulli Markov chains, it also does not depend on $\gamma_{l-1}^{(i)}$. ■

To state the lower bound for the system under consideration, the following definitions are used. Let $\gamma_l^{(\bar{i})}$ denote all the elements of γ_l except $\gamma_l^{(i)}$, and let $\gamma_l^{(\overline{i,j})}$ denote all the elements of γ_l except $\gamma_l^{(i)}$ and $\gamma_l^{(j)}$, for $i \neq j$. Using this notation, the definitions (22), shown at the bottom of the page, are introduced.

The application of the sequential WWLB to a fault-prone system satisfying Assumptions 3.1, 3.2, and 3.3 yields estimation error lower bounds for the state and fault vectors of the system. These lower bounds, which constitute the main results of the paper, are stated in the following Theorem.

Theorem 3.2: Consider a dynamic system satisfying Assumptions 3.1, 3.2, and 3.3. A WWLB for the estimation error covariance matrix of the state vector x_k is given by

$$E\left[\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)\left(x_{k+1} - \hat{x}_{k+1|k+1}\right)^T\right] \geqslant \left(J_{k+1}^x\right)^{-1} \tag{23}$$

where J_k^x is computed using the recursion

$$J_{k+1}^x = D_k^{22} - D_k^{21}\left(J_k^x + D_k^{11}\right)^{-1} D_k^{12} \tag{24a}$$

$$J_0^x = -E\left[\Delta_{x_0}^{x_0} \ln p_{x_0}(x_0)\right] \tag{24b}$$

and the matrices D_k^{ij} are given by

$$D_k^{11} = -E\left[\Delta_{x_k}^{x_k} \ln p_{x_{k+1}|x_k,\gamma_{k+1}}(x_{k+1} \mid x_k, \gamma_{k+1})\right] \tag{25a}$$

$$D_k^{12} = -E\left[\Delta_{x_k}^{x_{k+1}} \ln p_{x_{k+1}|x_k,\gamma_{k+1}}(x_{k+1} \mid x_k, \gamma_{k+1})\right] = {D_k^{21}}^T \tag{25b}$$

$$D_k^{22} = -E\left[\Delta_{x_{k+1}}^{x_{k+1}} \ln p_{x_{k+1}|x_k,\gamma_{k+1}}(x_{k+1} \mid x_k, \gamma_{k+1})\right] - E\left[\Delta_{x_{k+1}}^{x_{k+1}} \ln p_{y_{k+1}|x_{k+1},\gamma_{k+1}}(y_{k+1} \mid x_{k+1}, \gamma_{k+1})\right]. \tag{25c}$$

In equations (24b) and (25), the operator Δ_ξ^η is defined as

$$\Delta_\xi^\eta \triangleq \nabla_\xi \nabla_\eta^T \tag{26}$$

where

$$\nabla_\zeta \triangleq \left[\frac{\partial}{\partial \zeta_1} \quad \frac{\partial}{\partial \zeta_2} \quad \cdots \quad \frac{\partial}{\partial \zeta_n}\right]^T, \qquad \forall\, \zeta \in \mathbb{R}^n \tag{27}$$

is the gradient operator.

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{p_{x_l,\gamma_l|x_{l-1},\gamma_{l-1}}\left(X_l, \Gamma_l \mid X_{l-1}, \Gamma_{l-1} + \tilde{h}^{(i)}\right)} \\ &\quad \times \sqrt{p_{x_l,\gamma_l|x_{l-1},\gamma_{l-1}}\left(X_l, \Gamma_l \mid X_{l-1}, \Gamma_{l-1}\right)} dX_l d\Gamma_l \\ &\quad = \int_{-\infty}^{+\infty} \sqrt{p_{\gamma_l^{(i)}|\gamma_{l-1}^{(i)}}\left(\Gamma_l^{(i)} \mid \Gamma_{l-1}^{(i)} + 1\right) p_{\gamma_l^{(i)}|\gamma_{l-1}^{(i)}}\left(\Gamma_l^{(i)} \mid \Gamma_{l-1}^{(i)}\right)} d\Gamma_l^{(i)} \end{aligned} \tag{21}$$

$$\begin{aligned} \beta^{(i)}\left(x_{l-1}, \gamma_l^{(\bar{i})}\right) \triangleq& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\bar{i})}, \gamma_l^{(i)} = 1\right)} \\ &\times \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\bar{i})}, \gamma_l^{(i)} = 0\right)} dY_l dX_l \end{aligned} \tag{22a}$$

$$\begin{aligned} B^{(i,j)}\left(x_{l-1}, \gamma_l^{(\overline{i,j})}\right) \triangleq& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\overline{i,j})}, \gamma_l^{(i)} = 1, \gamma_l^{(j)} = 0\right)} \\ &\times \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\overline{i,j})}, \gamma_l^{(i)} = 0, \gamma_l^{(j)} = 1\right)} dY_l dX_l \end{aligned} \tag{22b}$$

$$\begin{aligned} \bar{B}^{(i,j)}\left(x_{l-1}, \gamma_l^{(\overline{i,j})}\right) \triangleq& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\overline{i,j})}, \gamma_l^{(i)} = 1, \gamma_l^{(j)} = 1\right)} \\ &\times \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\overline{i,j})}, \gamma_l^{(i)} = 0, \gamma_l^{(j)} = 0\right)} dY_l dX_l \end{aligned} \tag{22c}$$

$$\begin{aligned} c^{(i)}\left(x_{l-1}, \gamma_{l+1}^{(\bar{i})}, \gamma_l^{(\bar{i})}\right) \triangleq& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \beta^{(i)}\left(X_l, \gamma_{l+1}^{(\bar{i})}\right) \\ &\times \sqrt{p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\bar{i})}, \gamma_l^{(i)} = 1\right) p_{y_l,x_l|x_{l-1},\gamma_l}\left(Y_l, X_l \mid x_{l-1}, \gamma_l^{(\bar{i})}, \gamma_l^{(i)} = 0\right)} dY_l dX_l. \end{aligned} \tag{22d}$$

The corresponding WWLB for the estimation error covariance matrix of the fault vector γ_k is given by

$$E\left[\left(\gamma_{k+1}-\hat{\gamma}_{k+1|k+1}\right)\left(\gamma_{k+1}-\hat{\gamma}_{k+1|k+1}\right)^T\right] \geqslant \left(J^{\gamma}_{k+1}\right)^{-1} \tag{28}$$

where J^{γ}_k is computed using the recursion

$$\begin{aligned} J^{\gamma}_{k+1} =& G^{(k+1)}_{\gamma}(k+1,k+1) - G^{(k+1)}_{\gamma}(k+1,k) \\ &\times \left(J^{\gamma}_k + G^{(k+1)}_{\gamma}(k,k) - G^{(k)}_{\gamma}(k,k)\right)^{-1} \\ &\times G^{(k+1)}_{\gamma}(k,k+1) \end{aligned} \tag{29a}$$

$$J^{\gamma}_0 = G^{(0)}_{\gamma}(0,0). \tag{29b}$$

In (29), the elements of the matrix $G^{(k+1)}_{\gamma}(k+1,k+1)$ are given by $(i \neq j)$ are given by equation (30) at the bottom of the page, $G^{(k+1)}_{\gamma}(k+1,k)$ is a diagonal matrix with diagonal entries, as shown by equation (31) at the bottom of the page, and $G^{(k+1)}_{\gamma}(k,k) - G^{(k)}_{\gamma}(k,k)$ is a diagonal matrix with diagonal entries

$$G^{(k+1)}_{\gamma}(k,k)_{ii} - G^{(k)}_{\gamma}(k,k)_{ii} = \frac{\frac{1}{\left(\sqrt{P^{(i)}_{11}P^{(i)}_{10}}+\sqrt{\left(1-P^{(i)}_{11}\right)\left(1-P^{(i)}_{10}\right)}\right)^2} - 1}{\left(E\left[\beta^{(i)}\left(x_{k-1},\gamma^{(\bar{i})}_k\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_{k-1}}\left(1-P^{(i)}_{1,\gamma^{(i)}_{k-1}}\right)}\right]\right)^2}. \tag{32}$$

Proof: For the sake of brevity, only the main highlights of the proof are presented herein. The complete details of the proof can be found in [28].

The expressions (13) and (16), together with the symmetry Assumption 3.3, render all matrices $G^{(\cdot)}(\cdot,\cdot)$ block-diagonal, i.e.,

$$\begin{aligned} G^{(l)}(\xi,\eta) &= \begin{bmatrix} G^{(l)}_x(\xi,\eta) & 0 \\ 0 & G^{(l)}_{\gamma}(\xi,\eta) \end{bmatrix} \\ G^{(l)}_x(\xi,\eta) &\in \mathbb{R}^{n_x\times n_x}, \qquad G^{(l)}_{\gamma}(\xi,\eta) \in \mathbb{R}^{N\times N}. \end{aligned} \tag{33}$$

Equations (10b) and (33) render J_0 also a block-diagonal matrix, given by (24b) and (29b). Consequently, all matrices on the right-hand side (RHS) of (10a) are block-diagonal, rendering J_k a block-diagonal matrix for all time instances k. Thus, the lower bounds for x_k and γ_k can be computed separately by substituting the corresponding blocks of $G^{(\cdot)}(\cdot,\cdot)$ into (10a). Using a Taylor expansion, it can be shown that

$$G^{(k+1)}_x(k+1,k+1) = \varepsilon^2 D^{22}_k + o\left(\varepsilon^2\right) \tag{34a}$$

$$\begin{aligned} G^{(k+1)}_x(k,k+1) &= G^{(k+1)}_x(k+1,k)^T \\ &= \varepsilon^2 D^{12}_k + o\left(\varepsilon^2\right) \end{aligned} \tag{34b}$$

$$G^{(k+1)}_x(k,k) - G^{(k)}_x(k,k) = \varepsilon^2 D^{11}_k + o\left(\varepsilon^2\right) \tag{34c}$$

where the matrices D^{ij}_k are defined in (25). In addition, (30), (31), and (32) are derived in [28]. Finally, substituting these values of $G^{(\cdot)}_x(\cdot,\cdot)$ and $G^{(\cdot)}_{\gamma}(\cdot,\cdot)$ into (10a) completes the proof. ■

$$G^{(k+1)}_{\gamma}(k+1,k+1)_{ii} = \frac{1}{\left(E\left[\beta^{(i)}\left(x_k,\gamma^{(\bar{i})}_{k+1}\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_k}\left(1-P^{(i)}_{1,\gamma^{(i)}_k}\right)}\right]\right)^2} \tag{30a}$$

$$G^{(k+1)}_{\gamma}(k+1,k+1)_{ij} = 2\frac{E\left[\left(B^{(i,j)}\left(x_k,\gamma^{(\overline{i,j})}_{k+1}\right)-\bar{B}^{(i,j)}\left(x_k,\gamma^{(\overline{i,j})}_{k+1}\right)\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_k}\left(1-P^{(i)}_{1,\gamma^{(i)}_k}\right)}\sqrt{P^{(j)}_{1,\gamma^{(j)}_k}\left(1-P^{(j)}_{1,\gamma^{(j)}_k}\right)}\right]}{E\left[\beta^{(i)}\left(x_k,\gamma^{(\bar{i})}_{k+1}\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_k}\left(1-P^{(i)}_{1,\gamma^{(i)}_k}\right)}\right]E\left[\beta^{(j)}\left(x_k,\gamma^{(\bar{j})}_{k+1}\right)\sqrt{P^{(j)}_{1,\gamma^{(j)}_k}\left(1-P^{(j)}_{1,\gamma^{(j)}_k}\right)}\right]}$$
$$i \neq j \tag{30b}$$

$$\begin{aligned} G^{(k+1)}_{\gamma}(k+1,k)_{ii} =& 2\frac{\sqrt{P^{(i)}_{10}\left(1-P^{(i)}_{11}\right)}-\sqrt{P^{(i)}_{11}\left(1-P^{(i)}_{10}\right)}}{\sqrt{P^{(i)}_{11}P^{(i)}_{10}}+\sqrt{\left(1-P^{(i)}_{11}\right)\left(1-P^{(i)}_{10}\right)}} \\ &\times \frac{E\left[c^{(i)}\left(x_{k-1},\gamma^{(\bar{i})}_{k+1},\gamma^{(\bar{i})}_k\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_{k-1}}\left(1-P^{(i)}_{1,\gamma^{(i)}_{k-1}}\right)}\right]}{E\left[\beta^{(i)}\left(x_k,\gamma^{(\bar{i})}_{k+1}\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_k}\left(1-P^{(i)}_{1,\gamma^{(i)}_k}\right)}\right]E\left[\beta^{(i)}\left(x_{k-1},\gamma^{(\bar{i})}_k\right)\sqrt{P^{(i)}_{1,\gamma^{(i)}_{k-1}}\left(1-P^{(i)}_{1,\gamma^{(i)}_{k-1}}\right)}\right]} \end{aligned} \tag{31}$$

Remark 3.2: The lower bound for the continuously distributed state vector x_k, given by (23)–(27), is similar to the sequential version of the CRLB [13]. The only difference is that all the lower bound associated pdf's are conditioned on the fault vector γ_{k+1}.

Remark 3.3: The quantities defined in (22), which appear in the expressions for the lower bounds on the fault indicators, are of the general form

$$\int \sqrt{p_{a|s}(A \mid S_1) p_{a|s}(A \mid S_2)} dA.$$

Being closely related to the Bhattacharyya distance [12, p. 127], this form reflects the sensitivity of the distribution $p_{a|s}(A \mid S)$ to changes in the conditioning parameter s. In the present case, the distribution $p_{a|s}(A \mid S)$ is replaced by the distributions associated with the system model, and the conditioning is performed on the values of the fault indicators. Intuitively, it is clear that the higher the sensitivity to the fault indicators, the smaller are the achievable estimation errors and the lower are the bounds.

The computation of the blocks $G_x^{(\cdot)}(\cdot,\cdot)$ and $G_\gamma^{(\cdot)}(\cdot,\cdot)$ is not straightforward in the general case. However, several special cases of practical importance are considered in Section IV.

IV. White Fault Sequence

Suppose that the fault sequence $\{\gamma_k\}_{k=0}^{\infty}$ is white, i.e.,

$$P_{10}^{(i)}(k \mid k-1) = P_{11}^{(i)}(k \mid k-1) = p_k^{(i)} \quad \forall\, i = 1, 2, \ldots, N. \tag{35}$$

Then, the following Theorem can be stated.

Theorem 4.1: If the fault sequence $\{\gamma_k\}_{k=0}^{\infty}$ is white, the WWLB for the estimation error covariance matrix of γ_k is given by

$$E\left[\left(\gamma_{k+1} - \hat{\gamma}_{k+1|k+1}\right)\left(\gamma_{k+1} - \hat{\gamma}_{k+1|k+1}\right)^T\right] \geqslant G_\gamma^{(k+1)}(k+1, k+1)^{-1} \tag{36}$$

where the entries of $G_\gamma^{(k+1)}(k+1,k+1)$ are given by

$$G_\gamma^{(k+1)}(k+1,k+1)_{ii} = \frac{1}{p_{k+1}^{(i)}\left(1 - p_{k+1}^{(i)}\right)\left(E\left[\beta^{(i)}\left(x_k, \gamma_{k+1}^{(\bar{i})}\right)\right]\right)^2} \tag{37a}$$

$$G_\gamma^{(k+1)}(k+1,k+1)_{ij} = 2\frac{E\left[\left(B^{(i,j)}\left(x_k, \gamma_{k+1}^{(\overline{i,j})}\right) - \bar{B}^{(i,j)}\left(x_k, \gamma_{k+1}^{(\overline{i,j})}\right)\right)\right]}{E\left[\beta^{(i)}\left(x_k, \gamma_{k+1}^{(\bar{i})}\right)\right] E\left[\beta^{(j)}\left(x_k, \gamma_{k+1}^{(\bar{j})}\right)\right]} \tag{37b}$$

for $i, j = 1, 2, \ldots, N$ and $i \neq j$.

Proof: For a white fault sequence

$$\sqrt{P_{10}^{(i)}\left(1 - P_{11}^{(i)}\right)} - \sqrt{P_{11}^{(i)}\left(1 - P_{10}^{(i)}\right)} = 0 \tag{38}$$

yielding $G_\gamma^{(k+1)}(k+1,k) = 0$ according to (31). Together with (28) and (29a), this result yields (36). Equations (37a) and (37b) are obtained via substituting (35) into (30). ■

V. Bernoulli Markov Chain

Consider the case of a single Bernoulli Markov chain measured by a continuously distributed measurement. This case is described by the fault-prone system defined in Section III, where γ_k is a scalar and there is no continuously distributed state vector x_k, i.e.,

$$z_k = \gamma_k. \tag{39}$$

Theorem 5.1: For the Bernoulli Markov chain defined above and for the following selection of H:

$$H^{(k+1)} = I_{(k+2)\times(k+2)} \tag{40}$$

the sequential WWLB becomes

$$E\left[\left(\gamma_{k+1} - \hat{\gamma}_{k+1|k+1}\right)^2\right] \geqslant \frac{1}{J_{k+1}} \tag{41}$$

where J_{k+1} satisfies the recursion

$$J_{k+1} = G^{(k+1)}(k+1,k+1) - \frac{G^{(k+1)}(k+1,k)^2}{J_k + \left(G^{(k+1)}(k,k) - G^{(k)}(k,k)\right)}. \tag{42}$$

The expressions for $G^{(\cdot)}(\cdot,\cdot)$ are given by

$$G^{(k+1)}(k+1,k+1) = \frac{1}{\beta_{k+1}^2\left(E\left[\sqrt{P_{1,\gamma_k}\left(1 - P_{1,\gamma_k}\right)}\right]\right)^2} \tag{43a}$$

$$G^{(k+1)}(k,k) - G^{(k)}(k,k) = \left(\frac{1}{\left(\sqrt{P_{11}P_{10}} + \sqrt{(1-P_{11})(1-P_{10})}\right)^2} - 1\right) \times \frac{1}{\beta_k^2\left(E\left[\sqrt{P_{1,\gamma_{k-1}}\left(1 - P_{1,\gamma_{k-1}}\right)}\right]\right)^2} \tag{43b}$$

$$G^{(k+1)}(k+1,k) = 2\frac{\sqrt{P_{10}(1-P_{11})} - \sqrt{P_{11}(1-P_{10})}}{\left(\sqrt{P_{11}P_{10}} + \sqrt{(1-P_{11})(1-P_{10})}\right)} \times \frac{1}{E\left[\sqrt{P_{1,\gamma_k}\left(1 - P_{1,\gamma_k}\right)}\right]} \tag{43c}$$

and

$$\beta_l \triangleq \int_{-\infty}^{+\infty} \sqrt{p_{y_l|\gamma_l}(Y_l \mid 1) p_{y_l|\gamma_l}(Y_l \mid 0)} dY_l. \tag{44}$$

Proof: In the present case the terms $G_\gamma^{(\cdot)}(\cdot,\cdot)$ are scalar. It follows from (22a) that

$$\beta^{(1)}\left(x_{l-1}, \gamma_l^{(\bar{1})}\right) = \int_{-\infty}^{+\infty} \sqrt{p_{y_l|\gamma_l}(Y_l \mid 1) p_{y_l|\gamma_l}(Y_l \mid 0)} dY_l = \beta_l. \tag{45}$$

Similarly, (22d) yields

$$c^{(1)}\left(x_{l-1},\gamma_{l+1}^{(\bar{1})},\gamma_l^{(\bar{1})}\right)=\beta_{l+1}\beta_l. \tag{46}$$

The results (41) and (42) follow from (28) and (29a), respectively, whereas (43a)–(43c) are obtained by using (45) and (46) in (30a), (32), and (31). ■

Corollary 5.1: If $\{\gamma_k\}_{k=1}^{\infty}$ is a white sequence, then, according to Theorem 4.1

$$G^{(k+1)}(k+1,k)=0 \tag{47}$$

and the lower bound of Theorem 5.1 reduces to

$$\begin{aligned}E&\left[\left(\gamma_k-\hat{\gamma}_{k|k}\right)^2\right]\\ &\geqslant\frac{1}{G^{(k)}(k,k)}\\ &=p_k(1-p_k)\\ &\quad\times\left[\int_{-\infty}^{+\infty}\sqrt{p_{y_k|z_k}(Y_k\mid 1)p_{y_k|z_k}(Y_k\mid 0)}dY_k\right]^2\end{aligned} \tag{48}$$

where

$$p_k\triangleq P_{1i}(k\mid k-1). \tag{49}$$

Remark 5.1: The lower bound (48) has been reported earlier to be the WWLB on the estimation error variance of a Bernoulli random variable [23]. Here, γ_k plays the role of the estimated Bernoulli parameter, and y_k plays the role of the corresponding measurement vector. It is obvious that the rest of the measurement history, namely $\mathcal{Y}_{k-1}$, is irrelevant, since $\{\gamma_k\}_{k=1}^{\infty}$ is a white sequence.

VI. States Unaffected by Faults

Consider the case of a system where only the measurements are subjected to faults. In this case

$$p_{x_l|x_{l-1},\gamma_l}\left(X_l\mid X_{l-1},\Gamma_l\right)=p_{x_l|x_{l-1}}\left(X_l\mid X_{l-1}\right). \tag{50}$$

Define the quantities shown by equation (51) at the bottom of the page. The main result is now stated in the following Theorem.

Theorem 6.1: Consider a dynamic system satisfying property (50) and Assumptions 3.1, 3.2, and 3.3. Then, a WWLB on the state estimation covariance matrix is given by (23), (24a), (28), and (29a) for the following values of the matrices D_k^{ij}:

$$D_k^{11}=-E\left[\Delta_{x_k}^{x_k}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right] \tag{52a}$$

$$D_k^{12}=-E\left[\Delta_{x_k}^{x_{k+1}}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right]={D_k^{21}}^T \tag{52b}$$

$$\begin{aligned}D_k^{22}=&-E\left[\Delta_{x_{k+1}}^{x_{k+1}}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right]\\ &-E\left[\Delta_{x_{k+1}}^{x_{k+1}}\ln p_{y_{k+1}|x_{k+1},\gamma_{k+1}}(y_{k+1}\mid x_{k+1},\gamma_{k+1})\right]\end{aligned} \tag{52c}$$

and the following values of the elements of the matrices $G_\gamma^{(\cdot)}(\cdot,\cdot)$, as shown by equation (53) at the bottom of the next page.

Proof: The proof is omitted for conciseness. The interested reader is referred to [28] for its details. ■

A. Relation to the Cramér–Rao-Type Lower Bound of [22]

An interesting special case is obtained, if the state vector x_k is generated by a linear Gaussian system, i.e.,

$$x_{k+1}\mid x_k\sim\mathcal{N}\left(\Phi_{k+1}x_k,Q_{k+1}\right),\quad k=0,1,\ldots \tag{54a}$$

$$x_0\sim\mathcal{N}\left(0,\Sigma_0\right) \tag{54b}$$

and the system measurements, conditioned on the state and the fault vectors, are also Gaussian, i.e.,

$$y_{k+1}\mid x_{k+1},\gamma_{k+1}\sim\mathcal{N}\left(\Lambda(\gamma_{k+1})x_{k+1},R_{k+1}\right). \tag{55}$$

In this case, the lower bound can be evaluated in closed form, as stated in the next proposition.

Proposition 6.1: In a system satisfying (54) and (55), the sequential WWLB on the estimation error covariance matrix of the continuously distributed state vector x_k is given by

$$E\left[(x_k-\hat{x}_{k|k})(x_k-\hat{x}_{k|k})^T\right]\geqslant(J_k^x)^{-1} \tag{56}$$

$$\tilde{\beta}^{(i)}\left(x_l,\gamma_l^{(\bar{i})}\right)\triangleq\int_{-\infty}^{+\infty}\sqrt{p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\bar{i})},\gamma_l^{(i)}=1\right)p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\bar{i})},\gamma_l^{(i)}=0\right)}dY_l \tag{51a}$$

$$\widetilde{B}^{(i,j)}\left(x_l,\gamma_l^{(\overline{i,j})}\right)\triangleq\int_{-\infty}^{+\infty}\sqrt{p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\overline{i,j})},\gamma_l^{(i)}=1,\gamma_l^{(j)}=0\right)p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\overline{i,j})},\gamma_l^{(i)}=0,\gamma_l^{(j)}=1\right)}dY_l \tag{51b}$$

$$\widetilde{\widetilde{B}}^{(i,j)}\left(x_l,\gamma_l^{(\overline{i,j})}\right)\triangleq\int_{-\infty}^{+\infty}\sqrt{p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\overline{i,j})},\gamma_l^{(i)}=1,\gamma_l^{(j)}=1\right)p_{y_l|x_l,\gamma_l}\left(Y_l\mid x_l,\gamma_l^{(\overline{i,j})},\gamma_l^{(i)}=0,\gamma_l^{(j)}=0\right)}dY_l. \tag{51c}$$

where J_k^x is computed using the following recursion:

$$J_{k+1}^x = E\left[\Lambda(\gamma_{k+1})^T R_{k+1}^{-1}\Lambda(\gamma_{k+1})\right] + \left(\Phi_{k+1}\left(J_k^x\right)^{-1}\Phi_{k+1}^T + Q_{k+1}\right)^{-1} \tag{57a}$$

$$J_0^x = \Sigma_0^{-1}. \tag{57b}$$

Proof: For the system under consideration

$$D_k^{11} = -E\left[\Delta_{x_k}^{x_k}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right] = E\left[\Phi_{k+1}^T Q_{k+1}^{-1}\Phi_{k+1}\right] = \Phi_{k+1}^T Q_{k+1}^{-1}\Phi_{k+1} \tag{58a}$$

$$D_k^{12} = -E\left[\Delta_{x_k}^{x_{k+1}}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right] = E\left[\Phi_{k+1}^T Q_{k+1}^{-1}\right] = \Phi_{k+1}^T Q_{k+1}^{-1} \tag{58b}$$

$$\begin{aligned} D_k^{22} &= -E\left[\Delta_{x_{k+1}}^{x_{k+1}}\ln p_{x_{k+1}|x_k}(x_{k+1}\mid x_k)\right] - E\left[\Delta_{x_{k+1}}^{x_{k+1}}\ln p_{y_{k+1}|x_{k+1},\gamma_{k+1}}(y_{k+1}\mid x_{k+1},\gamma_{k+1})\right] \\ &= E\left[Q_{k+1}^{-1}\right] + E\left[\Lambda(\gamma_{k+1})^T R_{k+1}^{-1}\Lambda(\gamma_{k+1})\right] \\ &= E\left[\Lambda(\gamma_{k+1})^T R_{k+1}^{-1}\Lambda(\gamma_{k+1})\right] + Q_{k+1}^{-1} \end{aligned} \tag{58c}$$

$$J_0^x = -E\left[\Delta_{x_0}^{x_0}\ln p_{x_0}(x_0)\right] = \Sigma_0^{-1}. \tag{58d}$$

Substituting (58) into (24a) yields

$$\begin{aligned} J_{k+1}^x &= E\left[\Lambda(\gamma_{k+1})^T R_{k+1}^{-1}\Lambda(\gamma_{k+1})\right] + Q_{k+1}^{-1} - Q_{k+1}^{-1}\Phi_{k+1} \\ &\quad\times\left(J_k^x + \Phi_{k+1}^T Q_{k+1}^{-1}\Phi_{k+1}\right)^{-1}\Phi_{k+1}^T Q_{k+1}^{-1} \\ &= E\left[\Lambda(\gamma_{k+1})^T R_{k+1}^{-1}\Lambda(\gamma_{k+1})\right] + \left(\Phi_{k+1}(J_k^x)^{-1}\Phi_{k+1}^T + Q_{k+1}\right)^{-1}. \end{aligned} \tag{59}$$

■

The lower bound of Proposition 6.1 is identical to the CRLB-type lower bound recently presented in [22]. Notice, however, that the present result is more general as, unlike the systems considered in [22], the systems addressed herein are not restricted to homogenous irreducible Markov chains, and no assumptions are made on the structure of $\Lambda(\gamma_{k+1})$.

Remark 6.1: Unlike the result presented in Theorem 6.1, the corresponding lower bound reported in [22] on the estimation error covariance matrix of the fault vector is trivially zero. A nontrivial lower bound on the estimation error covariance matrix of the fault vector in a system satisfying (54) and (55) has been recently reported in [23]. However, unlike the result presented in Theorem 6.1, the lower bound of [23] is suitable only for systems where each measurement is affected by no more than one fault. Therefore, one of the benefits of the lower bound given in Theorem 6.1 is that it can be used to examine fault identifiability.

$$G_\gamma^{(k+1)}(k+1,k+1)_{ii} = \frac{1}{\left(E\left[\tilde\beta^{(i)}\left(x_{k+1},\gamma_{k+1}^{(\bar i)}\right)\right]\right)^2\left(E\left[\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1-P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right]\right)^2} \tag{53a}$$

$$G_\gamma^{(k+1)}(k+1,k+1)_{ij} = 2\frac{E\left[\widetilde B^{(i,j)}\left(x_{k+1},\gamma_{k+1}^{(\overline{i,j})}\right)\right] - E\left[\widetilde{\widetilde B}^{(i,j)}\left(x_{k+1},\gamma_{k+1}^{(\overline{i,j})}\right)\right]}{E\left[\tilde\beta^{(i)}\left(x_{k+1},\gamma_{k+1}^{(\bar i)}\right)\right]E\left[\tilde\beta^{(j)}\left(x_{k+1},\gamma_{k+1}^{(\bar j)}\right)\right]},\qquad i\neq j \tag{53b}$$

$$\begin{aligned} G_\gamma^{(k+1)}(k,k)_{ii} - G_\gamma^{(k)}(k,k)_{ii} &= \left(\frac{1}{\left(\sqrt{P_{11}^{(i)}P_{10}^{(i)}} + \sqrt{\left(1-P_{11}^{(i)}\right)\left(1-P_{10}^{(i)}\right)}\right)^2} - 1\right) \\ &\quad\times\frac{1}{\left(E\left[\tilde\beta^{(i)}\left(x_k,\gamma_k^{(\bar i)}\right)\right]\right)^2\left(E\left[\sqrt{P_{1,\gamma_{k-1}^{(i)}}^{(i)}\left(1-P_{1,\gamma_{k-1}^{(i)}}^{(i)}\right)}\right]\right)^2} \end{aligned} \tag{53c}$$

$$\begin{aligned} G_\gamma^{(k+1)}(k+1,k)_{ii} &= 2\frac{\sqrt{P_{10}^{(i)}\left(1-P_{11}^{(i)}\right)} - \sqrt{P_{11}^{(i)}\left(1-P_{10}^{(i)}\right)}}{\sqrt{P_{11}^{(i)}P_{10}^{(i)}} + \sqrt{\left(1-P_{11}^{(i)}\right)\left(1-P_{10}^{(i)}\right)}} \\ &\quad\times\frac{E\left[\tilde\beta^{(i)}\left(x_{k+1},\gamma_{k+1}^{(\bar i)}\right)\tilde\beta^{(i)}\left(x_k,\gamma_k^{(\bar i)}\right)\right]}{E\left[\tilde\beta^{(i)}\left(x_{k+1},\gamma_{k+1}^{(\bar i)}\right)\right]E\left[\tilde\beta^{(i)}\left(x_k,\gamma_k^{(\bar i)}\right)\right]E\left[\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1-P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right]} \end{aligned} \tag{53d}$$

$$G_\gamma^{(k+1)}(k,k)_{ij} - G_\gamma^{(k)}(k,k)_{ij} = G_\gamma^{(k+1)}(k+1,k)_{ij} = 0,\quad i\neq j. \tag{53e}$$

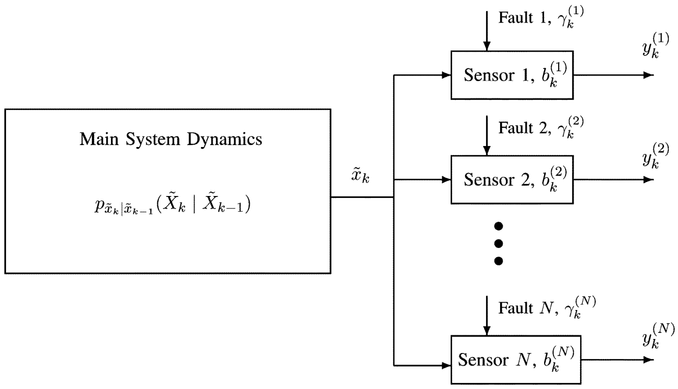

Fig. 1. Schematic description of a system with fault-free main dynamics and independent fault-prone measurement channels.

VII. Fault-Prone Measurement Channels

Suppose that the state vector x_k, defined in (12), and the measurement vector y_k, can be partitioned as

$$x_k = \left[\tilde{x}_k^T \quad {b_k^{(1)}}^T \quad {b_k^{(2)}}^T \quad \ldots \quad {b_k^{(N)}}^T\right]^T \tag{60a}$$

$$y_k = \left[{y_k^{(1)}}^T \quad {y_k^{(2)}}^T \quad \ldots \quad {y_k^{(N)}}^T\right]^T \tag{60b}$$

such that the following property holds:

$$\begin{aligned} &p_{y_l|x_l,\gamma_l}(Y_l \mid X_l, \Gamma_l)\, p_{x_l|x_{l-1},\gamma_l}(X_l \mid X_{l-1}, \Gamma_l) \\ &\times p_{\gamma_l|\gamma_{l-1}}(\Gamma_l \mid \Gamma_{l-1}) = p_{\tilde{x}_l|\tilde{x}_{l-1}}\left(\widetilde{X}_l \mid \widetilde{X}_{l-1}\right) \\ &\times \prod_{i=1}^{N} p_{y_l^{(i)}|\tilde{x}_l, b_l^{(i)} \gamma_l^{(i)}}\left(Y_l^{(i)} \mid \widetilde{X}_l, B_l^{(i)}, \Gamma_l^{(i)}\right) \\ &\times p_{b_l^{(i)}|b_{l-1}^{(i)}, \gamma_l^{(i)}}\left(B_l^{(i)} \mid B_{l-1}^{(i)}, \Gamma_l^{(i)}\right) \\ &\times p_{\gamma_l^{(i)}|\gamma_{l-1}^{(i)}}\left(\Gamma_l^{(i)} \mid \Gamma_{l-1}^{(i)}\right). \end{aligned} \tag{61}$$

Notice that the processes $\{\tilde{x}_k\}_{k=0}^{\infty}$, $\left\{b_k^{(1)}, \gamma_k^{(1)}\right\}_{k=0}^{\infty}$, $\left\{b_k^{(2)}, \gamma_k^{(2)}\right\}_{k=0}^{\infty}, \ldots, \left\{b_k^{(N)}, \gamma_k^{(N)}\right\}_{k=0}^{\infty}$ are mutually independent. This structure can be used to model a system with fault-free main dynamics and independent fault-prone measurement channels. The main dynamics of the system is characterized by the state vector $\tilde{x}_k$. The system has N independent measurement channels, whose outputs are $y_k^{(1)}$, $y_k^{(2)}, \ldots, y_k^{(N)}$, respectively. Each channel i is subjected to a single fault, indicated by $\gamma_k^{(i)}$, and its faulty behavior is characterized by a state vector $b_k^{(i)}$ (see Fig. 1). A computationally efficient estimation algorithm for this class of systems has been recently presented by the authors in [29].

Lemma 7.1: Consider a fault-prone dynamic system that satisfies (60) and (61). Then, the quantities $\beta^{(i)}\left(x_{l-1}, \gamma_l^{(\bar{i})}\right)$ and $c^{(i)}\left(x_{l-1}, \gamma_{l+1}^{(\bar{i})}, \gamma_l^{(\bar{i})}\right)$, defined in (22a) and (22d), are independent of $b_{l-1}^{(\bar{i})}$, $\gamma_l^{(\bar{i})}$ and $\gamma_{l+1}^{(\bar{i})}$, i.e.,

$$\beta^{(i)}\left(x_{l-1}, \gamma_l^{(\bar{i})}\right) = \beta^{(i)}\left(\tilde{x}_{l-1}, b_{l-1}^{(i)}\right) \tag{62a}$$

$$c^{(i)}\left(x_{l-1}, \gamma_{l+1}^{(\bar{i})}, \gamma_l^{(\bar{i})}\right) = c^{(i)}\left(\tilde{x}_{l-1}, b_{l-1}^{(i)}\right). \tag{62b}$$

In addition

$$B^{(i,j)}\left(x_{l-1}, \gamma_l^{(\overline{i,j})}\right) = \bar{B}^{(i,j)}\left(x_{l-1}, \gamma_l^{(\overline{i,j})}\right). \tag{63}$$

Proof: The result follows upon substituting (61) into (22). ■

Theorem 7.1: Consider a fault-prone dynamic system that satisfies (60) and (61). Then, the sequential WWLB for the fault vector γ_k, which is given in Theorem 3.2, becomes a set of separate lower bounds for each one of the elements of γ_k, i.e.,

$$E\left[\left(\gamma_{k+1}^{(i)} - \hat{\gamma}_{k+1|k+1}^{(i)}\right)^2\right] \geqslant \frac{1}{{J_{k+1}^{\gamma}}^{(i)}} \tag{64}$$

where ${J_{k+1}^{\gamma}}^{(i)}$ is computed using the recursion

$${J_{k+1}^{\gamma}}^{(i)} = g_{k+1,k+1}^{(i)} - \frac{\left(g_{k+1,k}^{(i)}\right)^2}{J_k^{\gamma(i)} + \Delta g_{k,k}^{(i)}} \tag{65a}$$

$$J_0^{\gamma(i)} = g_{0,0}^{(i)} \tag{65b}$$

and the terms $g_{k+1,k+1}^{(i)}$, $g_{k+1,k}^{(i)}$, and $\Delta g_{k,k}^{(i)}$ are calculated using expression (66), shown at the bottom of the next page.

Proof: By Assumption 3.1, the elements of the fault vector γ_0 are independent, rendering $G_{\gamma}^{(0)}(0,0)$ a diagonal matrix. Therefore, the matrix J_0^{γ} is also diagonal. Substituting (63) into (30b) renders $G_{\gamma}^{(k+1)}(k+1,k+1)$ diagonal too. In addition, the matrices $G_{\gamma}^{(k+1)}(k+1,k)$ and $G_{\gamma}^{(k+1)}(k,k) - G_{\gamma}^{(k)}(k,k)$ are, by Theorem 3.2, also diagonal. Consequently, all matrices on the RHS of (29a) are diagonal, rendering J_k^{γ} a diagonal matrix for all time instances k. Thus, the lower bounds for $\gamma_k^{(i)}$ can be computed separately by substituting the corresponding diagonal elements of the matrices $G_{\gamma}^{(\cdot)}(\cdot,\cdot)$ into (29a). Finally, equations (66) follow upon substituting (62) into the expressions (30a), (32), and (31). ■

Two interesting special cases are considered in the sequel.

A. Additive Faults

An interesting special case arises when the measurement faults have an "additive" nature, i.e.,

$$\begin{aligned} &p_{y_l^{(i)}|\tilde{x}_l, b_l^{(i)}, \gamma_l^{(i)}}\left(Y_l^{(i)} \mid \widetilde{X}_l, B_l^{(i)}, \Gamma_l^{(i)}\right) \\ &\quad = f\left(Y_l^{(i)} - \lambda_x\left(\widetilde{X}_l\right) - \lambda_b\left(B_l^{(i)}, \Gamma_l^{(i)}\right)\right). \end{aligned} \tag{67}$$

In this case, the following result holds.

Theorem 7.2: Consider a fault-prone dynamic system that satisfies (60), (61), and (67). Then

$$\beta^{(i)}\left(\tilde{x}_{l-1}, b_{l-1}^{(i)}\right) = \beta^{(i)}\left(b_{l-1}^{(i)}\right) \tag{68a}$$

$$c^{(i)}\left(\tilde{x}_{l-1}, b_{l-1}^{(i)}\right) = c^{(i)}\left(b_{l-1}^{(i)}\right). \tag{68b}$$

Proof: The result follows upon substituting (67) into (22). ■

Corollary 7.1: In a fault-prone dynamic system that satisfies (60), (61), and (67), the lower bounds on the estimation error variances of the fault indicators $\gamma_k^{(i)}$, given by Theorem 7.1, are unaffected by the distribution of $\tilde{x}_k$. In other words, these lower bounds can be calculated separately, disregarding the particular behavior of the state vector $\tilde{x}_k$.

B. Relation to the Lower Bound for Fault Indicators of [23]

Another special case is now considered, where the system of interest satisfies both the properties (60), (61), and the assumptions made in Section VI, namely, (50), (54), and (55). In other words, in the addressed system the states $b_k^{(i)}$ are absent, the state vector $\tilde{x}_k$ is generated by a linear Gaussian system, i.e.,

$$\tilde{x}_{k+1} \mid \tilde{x}_k \sim \mathcal{N}\left(\Phi_{k+1}\tilde{x}_k, Q_{k+1}\right) \tag{69}$$

and the system measurements, conditioned on the state and the fault vectors, are also Gaussian, i.e.,

$$y_{k+1}^{(i)} \mid \tilde{x}_{k+1}, \gamma_{k+1}^{(i)} \sim \mathcal{N}\left(\Lambda_{k+1}^{(i)}\left(\gamma_{k+1}^{(i)}\right)\tilde{x}_{k+1}, R_{k+1}^{(i)}\left(\gamma_{k+1}^{(i)}\right)\right). \tag{70}$$

This case has been recently addressed in [23], where nontrivial lower bounds on the estimation error variances of the fault indicators have been proposed. These lower bounds have the following form:

$$E\left[\left(\gamma_{k+1}^{(i)} - \hat{\gamma}_{k+1|k+1}^{(i)}\right)^2\right] \geqslant E\left[\tilde{\beta}^{(i)}\left(\tilde{x}_{k+1}\right)^2\right] \times E\left[P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)\right], \quad i = 1, 2, \ldots, N. \tag{71}$$

Comparing (71) to the lower bounds presented in Theorem 7.1, one can observe that the results are similar but not identical. Moreover, each of them can be tighter than the other depending on the particular system. Thus, in a system with white fault sequences, the lower bound presented in [23] is tighter than the lower bound given by Theorem 7.1. To demonstrate this fact notice that, by Theorems 4.1, 6.1, and 7.1, the latter lower bound is

$$E\left[\left(\gamma_{k+1}^{(i)} - \hat{\gamma}_{k+1|k+1}^{(i)}\right)^2\right] \geqslant \frac{1}{g_{k+1,k+1}^{(i)}} \tag{72}$$

where

$$\frac{1}{g_{k+1,k+1}^{(i)}} = \left(E\left[\tilde{\beta}^{(i)}\left(\tilde{x}_{k+1}\right)\right]\right)^2 \times \left(E\left[\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right]\right)^2 \tag{73}$$

and, by the Jensen inequality

$$\left(E\left[\tilde{\beta}^{(i)}\left(\tilde{x}_{k+1}\right)\right]\right)^2 \left(E\left[\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right]\right)^2 \leqslant E\left[\tilde{\beta}^{(i)}\left(\tilde{x}_{k+1}\right)^2\right] E\left[P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)\right]. \tag{74}$$

On the other hand, if the system under consideration does not depend on the state vector $\tilde{x}_k$, and, in addition

$$P_{11}^{(i)}\left(1 - P_{11}^{(i)}\right) = P_{10}^{(i)}\left(1 - P_{10}^{(i)}\right) \tag{75}$$

then the lower bound presented in [23] becomes

$$E\left[\tilde{\beta}^{(i)}\left(\tilde{x}_{k+1}\right)^2\right] E\left[P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)\right] = {\beta^{(i)}}^2 P_{11}^{(i)}\left(1 - P_{11}^{(i)}\right) = \frac{1}{g_{k+1,k+1}^{(i)}} \tag{76}$$

whereas the lower bound given by Theorem 7.1 takes the form

$$\frac{1}{g_{k+1,k+1}^{(i)} - \frac{\left(g_{k+1,k}^{(i)}\right)^2}{J_k^{\gamma(i)} + \Delta g_{k,k}^{(i)}}} \geqslant \frac{1}{g_{k+1,k+1}^{(i)}}. \tag{77}$$

$$g_{k+1,k+1}^{(i)} = \frac{1}{\left(E\left[\beta^{(i)}\left(\tilde{x}_k, b_k^{(i)}\right)\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right]\right)^2} \tag{66a}$$

$$\Delta g_{k,k}^{(i)} = \frac{\dfrac{1}{\left(\sqrt{P_{11}^{(i)}P_{10}^{(i)}} + \sqrt{\left(1-P_{11}^{(i)}\right)\left(1-P_{10}^{(i)}\right)}\right)^2} - 1}{\left(E\left[\beta^{(i)}\left(\tilde{x}_{k-1}, b_{k-1}^{(i)}\right)\sqrt{P_{1,\gamma_{k-1}^{(i)}}^{(i)}\left(1 - P_{1,\gamma_{k-1}^{(i)}}^{(i)}\right)}\right]\right)^2} \tag{66b}$$

$$g_{k+1,k}^{(i)} = 2\frac{\sqrt{P_{10}^{(i)}\left(1 - P_{11}^{(i)}\right)} - \sqrt{P_{11}^{(i)}\left(1 - P_{10}^{(i)}\right)}}{\sqrt{P_{11}^{(i)}P_{10}^{(i)}} + \sqrt{\left(1 - P_{11}^{(i)}\right)\left(1 - P_{10}^{(i)}\right)}} \times \frac{E\left[c^{(i)}\left(\tilde{x}_{k-1}, b_{k-1}^{(i)}\right)\sqrt{P_{1,\gamma_{k-1}^{(i)}}^{(i)}\left(1 - P_{1,\gamma_{k-1}^{(i)}}^{(i)}\right)}\right]}{E\left[\beta^{(i)}\left(\tilde{x}_k, b_k^{(i)}\right)\sqrt{P_{1,\gamma_k^{(i)}}^{(i)}\left(1 - P_{1,\gamma_k^{(i)}}^{(i)}\right)}\right] E\left[\beta^{(i)}\left(\tilde{x}_{k-1}, b_{k-1}^{(i)}\right)\sqrt{P_{1,\gamma_{k-1}^{(i)}}^{(i)}\left(1 - P_{1,\gamma_{k-1}^{(i)}}^{(i)}\right)}\right]}. \tag{66c}$$

VIII. Conclusion

This paper addresses applications of the sequential Weiss–Weinstein lower bound, presented by the authors in the companion paper, to fault-prone systems. In these systems, the importance of the bound stems from its use to infer about fault detectability and identifiability, to optimally configure sensor systems, and to provide an absolute figure of merit for the performance of fault-tolerant estimation algorithms. The bound is applied first to a general fault-prone system. Then, several special cases are considered. It is shown that several recently reported sequential lower bounds, derived for fault-prone systems, are special cases of, or else are closely related to, the new lower bound presented in the companion paper. It should be noted also that the proposed applications are not unique, as other bounds can be obtained for different selections of the Weiss–Weinstein lower bound free (tuning) parameters.

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