The signal detection problem described above yields the joint pdfs for r_1 and θ_1 according to (23) and (24) under H_0 and H_1, respectively. For the case when $\Delta\omega/\omega_c < 1$ the narrow-band approximation holds almost perfectly which results in a joint pdf with hardly any θ_1 dependence. When $\Delta\omega/\omega_c$ decreases further, the marginal pdfs of r_1 will asymptotically approach Rayleigh and Rice under H_0 and H_1, respectively. This is depicted in Figs. 4 and 5, where the joint pdf for r_1 and θ_1 under H_0 and H_1, respectively, is presented for $d = 3$ and $\Delta\omega/\omega_c = 0.6$. For the case when $d = 3$ and $\Delta\omega/\omega_c > 1$, the narrow-band approximation is not satisfied, which is manifested in a θ_1 dependence. In Figs. 6 and 7, the joint pdfs for r_1 and θ_1 under H_0 and H_1, respectively, for $d = 4$ and $\Delta\omega/\omega_c = 2$ show that the θ_1 dependence is evident under both hypotheses. By studying Figs. 6 and 7, it is clear that the probability of signals with θ_1 in the region of $-\pi/2$ or $\pi/2$ is dominating.

This leads to the conclusion, see (12), that the in-phase component A_c is contributing the most toward detecting the signals, which explains why the P_E^* curves in Fig. 2 converge to the same value for large bandwidth ratios, i.e., $\Delta\omega/\omega_c > 5$.

V. Conclusions

An analytical series expansion solution to the noncoherent detection problem has been presented. The resulting detector shows a significant advantage over the conventional noncoherent detector in the case of signal detection when the classical narrow-band approximation does not hold. This was illustrated by means of a transient signal detection example. In addition to the analytical solution presented, two pdfs for the in-phase and quadrature components, jointly expressed in polar coordinates, were derived. The radial marginal of these pdfs reduces to the classical Rayleigh and Rice pdf when the narrow-band approximation holds.

The application of this new detector is still to be further developed but since the noncoherent detector has been and is still widely used [6], [7], the potential need for an analytical solution to the noncoherent detection problem could be widespread, especially in the areas of radar, sonar, and nondestructive testing where the narrow-band approximation is not always satisfied.

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A New Estimation Error Lower Bound for Interruption Indicators in Systems With Uncertain Measurements

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***Abstract*—Optimal mean-square error estimators of systems with interrupted measurements are infinite dimensional, because these systems belong to the class of hybrid systems. This renders the calculation of a lower bound for the estimation error of the interruption process in these systems of particular interest. Recently it has been shown that a Cramér–Rao-type lower bound on the interruption process estimation error is trivially zero. In the present work, a nonzero lower bound for a class of systems with Markovian interruption variables is proposed. Derivable using the well-known Weiss–Weinstein bound, this lower bound can be easily evaluated using a simple recursive algorithm. The proposed lower bound is shown to depend on a measure of the interruption chain transitional determinism, the measurement noise sensitivity to interruption process switchings, and a measure of the system's state estimability. In some cases, identified in this correspondence, the proposed bound is tight. The use of the lower bound is illustrated via a simple numerical example.**

***Index Terms*—Estimation error lower bound, fault detection and isolation, hybrid systems.**

I. Introduction

Modern multisensor applications, such as navigation and target tracking systems, require the fusion of data acquired by a large number of different sensors. In many situations these sensors might be subjected to faults, either due to internal malfunctions, or because of external interferences. Typical examples of such scenarios include intended (hostile) global positioning system (GPS) jamming and spoofing, rate gyro misalignment due to input accelerations in improperly balanced inertial navigation systems, and magnetometer measurement errors, induced by magnetism-generating devices in the

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vicinity of the sensor. In view of present-day systems' high accuracy requirements, the problem of fault-tolerant filtering in multisensor systems is of major importance.

A popular approach to the modeling and analysis of systems with fault-prone sensors is based on using the framework of hybrid systems, or systems with switching parameters [1]. In this approach, one of the switching parameter values corresponds to the nominal system operation, whereas the others represent various fault situations [2], [3].

In systems with independent fault-prone sensors and fault-free dynamics, such as GPS receivers, the aforementioned model can be simplified: the faults in different measurement channels can be modeled as separate Markovian Bernoulli random processes, where "1" stands for a fault situation and "0" stands for no fault situation. Since the state vector is free of faults, these fault indicators, which are also referred to as interruption variables [4], affect only the system measurements.

It is well known that the optimal mean-square error filtering algorithm for hybrid systems, that provides the estimates of the state vector and the switching parameters, requires infinite computation resources [5]. Therefore, a variety of suboptimal techniques was proposed [1], [4]–[8]. Since the estimates of the interruption variables are suboptimal, it is of particular interest to obtain some measure of their efficiency, e.g., a lower bound on their estimation error. The most popular bound is the well-known Cramér–Rao lower bound (CRLB) [9, p. 84]. Unfortunately, this bound cannot be directly calculated for the estimation error of the interruption process, because the distribution of the interruption variables is discrete and, therefore, does not satisfy the CRLB's regularity conditions [9, p. 72].

A CRLB-type lower bound for a class of systems with fault-prone measurements has been recently presented in [10]. Lower bounds for both the state and the Markovian interruption variables of the system were derived, based on the sequential version of the CRLB for general nonlinear systems [11]. To facilitate the calculation of the CRLB for this class of systems, the discrete distribution was approximated by a continuous one and the lower bound was obtained via a limiting process applied to the approximating system. The results of [10] facilitate a relatively simple calculation of a nontrivial lower bound for the state vector of systems with fault-prone measurements. However, these results also indicate that the CRLB-type lower bound for the interruption process variables is trivially zero.

Unlike the CRLB, the Weiss–Weinstein lower bound [12] is essentially free from regularity conditions and, therefore, can be applied to systems with interrupted measurements. However, this bound requires to process the data in a batch form, rendering its application to dynamic systems rather cumbersome.

The present work derives a nonzero lower bound on the interruption variable estimation errors in systems with independent fault-prone sensors and fault-free dynamics. The relation between this lower bound and the Weiss–Weinstein bound is discussed. The proposed lower bound can be evaluated using a simple recursive algorithm, and is shown to depend on 1) a measure of the interruption chain transitional determinism, 2) the measurement noise sensitivity to interruption process switchings, and 3) a measure of the system's state estimability. In some cases, identified in this correspondence, this lower bound is tight.

The remainder of this correspondence is organized as follows. The system model is defined in Section II. Several preliminary results are presented in Section III. The main result of this correspondence, namely, the lower bound on the interruption process estimation error variances, is presented and derived in Section IV and then discussed in Section V. A simple numerical example illustrating the computation and use of the proposed lower bound is presented in Section VI. Concluding remarks are offered in the last section. For presentation clarity, the notational convention of [9] is adopted, according to which lower case and upper case letters are used to denote random variables and their realizations, respectively.

II. Problem Formulation

Consider the system with the following dynamics:

$$x_k = \Phi_k x_{k-1} + G_k w_k, \qquad x \in \mathbb{R}^n, \ k = 1, 2, \ldots \tag{1}$$

and the following measurement model:

$$y_k = \left[y_k^{(1)T}, y_k^{(2)T}, \ldots, y_k^{(N)T} \right]^T, \qquad y_k^{(i)} \in \mathbb{R}^{m_i}, \ i = 1, 2, \ldots, N \tag{2}$$

where $y_k^{(i)}$, which denotes the measurement of the ith measurement channel, is given, in the most general form, by

$$y_k^{(i)} = \widetilde{H}_k^{(i)}(\gamma_k^{(i)}) x_k + v_k^{(i)}, \qquad i = 1, 2, \ldots, N. \tag{3}$$

Here $\{w_k\}$ and $\{v_k^{(i)}\}$ are Gaussian white sequences with

$$w_k \sim \mathcal{N}\Big(0, Q_k\Big) \quad \text{and} \quad v_k^{(i)} \sim \mathcal{N}\Big(0, R_k^{(i)}(\gamma_k^{(i)})\Big)$$

where $R_k^{(i)}(\gamma_k^{(i)}) > 0$. The initial state is a random vector satisfying $x_0 \sim \mathcal{N}(0, \Sigma_0)$. Each interruption process $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ is assumed to be a Markovian Bernoulli sequence, i.e., taking values of 0 and 1, with the following initial and transition probabilities:

$$\Pr\{\gamma_0^{(i)} = 1\} = p_0^{(i)} \tag{4a}$$

$$\Pr\{\gamma_k^{(i)} = 1 \mid \gamma_{k-1}^{(i)} = j\} = P_{1j}^{(i)}(k \mid k-1), \qquad j \in \{0, 1\}. \tag{4b}$$

Since $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ is a Bernoulli sequence, (3) can be rewritten without loss of generality in the following form, that will be used in the sequel:

$$y_k^{(i)} = \left(H_k^{(i)} + \gamma_k^{(i)} \Delta H_k^{(i)} \right) x_k + v_k^{(i)}, \qquad i = 1, 2, \ldots, N \tag{5}$$

where

$$H_k^{(i)} \triangleq \widetilde{H}_k^{(i)}(0) \tag{6a}$$

$$\Delta H_k^{(i)} \triangleq \widetilde{H}_k^{(i)}(1) - \widetilde{H}_k^{(i)}(0). \tag{6b}$$

The noise sequences $\{w_k\}_{k=1}^{\infty}$ and $\{v_k^{(i)}\}_{k=1}^{\infty}$, the initial state vector x_0, and the interruption sequences $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ are assumed to be mutually independent. For notational simplicity, the explicit time dependence is suppressed in the sequel in all places where it is clear by context.

The model defined above is applicable to a wide class of systems in the area of fault detection and isolation. In these systems, the state vector x_k comprises two parts: the main part is associated with the system dynamics and the secondary part is associated with the dynamics of the sensors when sensor faults occur. These fault states can describe various kinds of sensors' faulty behavior, e.g., measurement biases, or additive faulty measurement noises (white or colored). The interruption variables $\gamma_k^{(i)}$ play a role of fault indicators. If the system measurements are acquired at a low rate it can be also assumed that the fault indicators at different time instants are independent.

The following definitions will be used in the sequel:

$$\mathcal{Y}_k \triangleq \left[y_1^T, y_2^T, \ldots, y_k^T \right]^T \tag{7}$$

$$\gamma_k \triangleq \left[\gamma_k^{(1)}, \ldots, \gamma_k^{(N)} \right]. \tag{8}$$

Also, corresponding to (4a), the probability that the interruption process of channel i takes the value 1 at time k is defined as

$$p_k^{(i)} \triangleq \Pr\left\{ \gamma_k^{(i)} = 1 \right\}. \tag{9}$$

In addition, denote by $\hat{\gamma}_{k|k}^{(i)}$ any estimate of $\gamma_k^{(i)}$ based on measurements up to and including time k, and let Σ_k denote the covariance matrix of the state vector x_k.

The goal of this work is to derive a lower bound on the estimation error variances of the interruption processes $\{\gamma_k^{(i)}\}_{k=1}^{\infty}$, $i = 1, 2, \ldots, N$.

III. Preliminary Results

Before stating the main result of this correspondence, the following lemma is stated, proved, and discussed. The following definitions are used in the sequel. Let γ be a Bernoulli random variable with $\Pr\{\gamma = 1\} = p$ and let z be a measurement vector with conditional probability density functions (pdf) $p_{z|\gamma}(Z \mid i)$, $i \in \{0, 1\}$. Then, $\hat{\gamma}(z)$ denotes any estimate of γ based on the measurements z and $\hat{\gamma}_{\rm opt}$ denotes the mean-square optimal estimate, i.e.,

$$\hat{\gamma}_{\rm opt} \triangleq E[\gamma \mid z]. \tag{10}$$

Finally, for notation simplicity, define

$$f_i \triangleq p_{z|\gamma}(Z \mid i), \qquad i \in \{0, 1\}. \tag{11}$$

Lemma 3.1:

$$E\left[(\hat{\gamma}(z) - \gamma)^2\right] \geqslant p(1-p) \left[\int_{-\infty}^{+\infty} \sqrt{p_{z|\gamma}(Z \mid 0) p_{z|\gamma}(Z \mid 1)} \mathrm{d}Z\right]^2. \tag{12}$$

Proof: Since the conditional expectation is the optimal estimate in the mean-square sense

$$E\left[(\hat{\gamma}(z) - \gamma)^2\right] \geqslant E\left[(\hat{\gamma}_{\rm opt} - \gamma)^2\right]. \tag{13}$$

Now, using the smoothing property of the conditional expectation and noting that γ is a Bernoulli random variable yields

$$E\left[(\hat{\gamma}_{\rm opt} - \gamma)^2\right] = E\left[E\left[(\hat{\gamma}_{\rm opt} - \gamma)^2 \mid z\right]\right] = E\left[\hat{\gamma}_{\rm opt} - \hat{\gamma}_{\rm opt}^2\right]. \tag{14}$$

On the other hand,

$$\hat{\gamma}_{\rm opt} = E[\gamma \mid z] = \Pr\{\gamma = 1 \mid z\} = \frac{f_1 p}{f_1 p + f_0(1-p)} \tag{15}$$

so that

$$\begin{aligned}\hat{\gamma}_{\rm opt} - \hat{\gamma}_{\rm opt}^2 &= \frac{f_1 p}{f_1 p + f_0(1-p)} - \frac{f_1^2 p^2}{[f_1 p + f_0(1-p)]^2} \\ &= p(1-p) \frac{f_1 f_0}{[f_1 p + f_0(1-p)]^2}.\end{aligned} \tag{16}$$

Therefore,

$$E\left[\hat{\gamma}_{\rm opt} - \hat{\gamma}_{\rm opt}^2\right] = p(1-p) E\left[\frac{f_1 f_0}{[f_1 p + f_0(1-p)]^2}\right]. \tag{17}$$

Notice that in the above expressions f_1 and f_0 are functions of the random vector z. Now

$$\begin{aligned}E\left[\frac{f_1 f_0}{[f_1 p + f_0(1-p)]^2}\right] &= \int_{-\infty}^{+\infty} \frac{f_1 f_0}{[f_1 p + f_0(1-p)]^2} p_z(Z) \mathrm{d}Z \\ &= \int_{-\infty}^{+\infty} \frac{f_1 f_0}{f_1 p + f_0(1-p)} \mathrm{d}Z.\end{aligned} \tag{18}$$

Recall also that

$$\int_{-\infty}^{+\infty} (f_1 p + f_0(1-p)) \mathrm{d}Z = \int_{-\infty}^{+\infty} p_z(Z) \mathrm{d}Z = 1. \tag{19}$$

Using the Cauchy–Schwartz inequality yields

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{f_1 f_0}{f_1 p + f_0(1-p)} \mathrm{d}Z \\ &= \int_{-\infty}^{+\infty} \left|\sqrt{\frac{f_1 f_0}{f_1 p + f_0(1-p)}}\right|^2 \mathrm{d}Z \int_{-\infty}^{+\infty} \left|\sqrt{f_1 p + f_0(1-p)}\right|^2 \mathrm{d}Z \\ &\geqslant \left|\int_{-\infty}^{+\infty} \sqrt{\frac{f_1 f_0}{f_1 p + f_0(1-p)}} \sqrt{f_1 p + f_0(1-p)} \mathrm{d}Z\right|^2 \\ &= \left[\int_{-\infty}^{+\infty} \sqrt{f_1 f_0} \mathrm{d}Z\right]^2 = \left[\int_{-\infty}^{+\infty} \sqrt{p_{z|\gamma}(Z \mid 1) p_{z|\gamma}(Z \mid 0)} \mathrm{d}Z\right]^2\end{aligned} \tag{20}$$

and the Lemma follows upon combining (13), (14), (17), (18), and (20). □

Remark 3.1: The lower bound presented in Lemma 3.1 can be alternatively written as

$$\sqrt{E\left[(\hat{\gamma}(z) - \gamma)^2\right]} \geqslant \frac{\sqrt{\mathrm{var}[\gamma]}}{e^d} \tag{21}$$

where

$$d \triangleq -\ln \int_{-\infty}^{+\infty} \sqrt{p_{z|\gamma}(Z \mid 0) p_{z|\gamma}(Z \mid 1)} \mathrm{d}Z \tag{22}$$

is the Bhattacharyya distance between the distributions $p_{z|\gamma}(Z \mid 0)$ and $p_{z|\gamma}(Z \mid 1)$ [9, p. 127]. Note also that the square root of the right-hand side (RHS) of (12) is known in the literature as the Matushita error [13, p. 23].

Corollary 3.1: The optimal estimator defined in (10) is efficient with respect to the lower bound given in (12) iff one of the following conditions is satisfied:

$$p_{z|\gamma}(Z \mid 0) p_{z|\gamma}(Z \mid 1) = 0, \qquad \forall Z \tag{23a}$$

$$p_{z|\gamma}(Z \mid 0) = p_{z|\gamma}(Z \mid 1), \qquad \forall Z. \tag{23b}$$

Proof: The equality in (20) (see the proof of Lemma 3.1) exists iff there exists a number α such that

$$\frac{f_1 f_0}{f_1 p + f_0(1-p)} = \alpha[f_1 p + f_0(1-p)], \qquad \forall Z. \tag{24}$$

Setting $\alpha = 0$ yields the condition (23a). For $\alpha \neq 0$ (24) gives

$$p^2 \alpha f_1^2 + 2 f_0 \Big[p(1-p)\alpha - 0.5\Big] f_1 + (1-p)^2 \alpha f_0^2 = 0. \tag{25}$$

This equality holds only if

$$\alpha p(1-p) \leqslant \frac{1}{4} \tag{26}$$

in which case the following relations are obtained:

$$f_1 = \frac{f_0}{p^2\alpha}\left[\frac{1}{2} - \alpha p(1-p) \pm \sqrt{\frac{1}{4} - \alpha p(1-p)}\right] \tag{27}$$

if $p \neq 0$ and

$$f_1 = \alpha f_0 \tag{28}$$

if $p = 0$. In both cases, f_1 is proportional to f_0. Finally, condition (23b) follows from the fact that both f_0 and f_1 are pdf's.

Remark 3.2: The result presented in the lemma is, in fact, a special case of the well-known Weiss–Weinstein bound. To show this, recall that the general form of this bound is [12]

$$E\left[(\gamma - \hat{\gamma}(z))^2\right] \geqslant \frac{E[\gamma\psi(z,\gamma)]^2}{E\left[\psi^2(z,\gamma)\right]} \tag{29}$$

where

$$\psi(z,\gamma) \triangleq \begin{cases} L^s(z;\gamma+h,\gamma) - L^{1-s}(z;\gamma-h,\gamma), & \gamma \in \mathcal{G}' \\ 0, & \gamma \notin \mathcal{G}'. \end{cases} \tag{30}$$

In (30), the likelihood function $L(z;\gamma_1,\gamma_2)$ is defined as

$$L(Z;\Gamma_1,\Gamma_2) \triangleq \frac{p_{z,\gamma}(Z,\Gamma_1)}{p_{z,\gamma}(Z,\Gamma_2)} \tag{31}$$

and the set $\mathcal{G}'$ is defined as

$$\mathcal{G}' \triangleq \{\Gamma \mid p_{z,\gamma}(z,\Gamma) > 0 \quad \text{a.e.} \quad z \in \mathbb{R}\}. \tag{32}$$

Now let $h = 1$, $s = 0.5$, and

$$p_{z,\gamma}(Z,\Gamma) = p_{z|\gamma}(Z \mid \Gamma)((1-p)\delta(\Gamma) + p\delta(\Gamma - 1)) \tag{33}$$

where $\delta(\Gamma)$ is Dirac's delta function. This assumption yields $\mathcal{G}' = \{0,1\}$. Moreover

$$\begin{aligned} E[\gamma\psi(z,\gamma)] &= \int_{-\infty}^{+\infty} \mathrm{d}Z \int_{\mathcal{G}'} \Gamma\psi(Z,\Gamma)p_{z,\gamma}(Z,\Gamma)\mathrm{d}\Gamma \\ &= \int_{-\infty}^{+\infty} \mathrm{d}Z \int_{-\infty}^{+\infty} \Gamma\Big[p^s_{z,\gamma}(Z,\Gamma+h)p^{1-s}_{z,\gamma}(Z,\Gamma) \\ &\quad - p^s_{z,\gamma}(Z,\Gamma)p^{1-s}_{z,\gamma}(Z,\Gamma-h)\Big]\mathrm{d}\Gamma \\ &= \int_{-\infty}^{+\infty} \mathrm{d}Z \int_{-\infty}^{+\infty} (\Gamma+h)p^s_{z,\gamma}(Z,\Gamma+h)p^{1-s}_{z,\gamma}(Z,\Gamma)\mathrm{d}\Gamma \\ &\quad - \int_{-\infty}^{+\infty} \mathrm{d}Z \int_{-\infty}^{+\infty} \Gamma p^s_{z,\gamma}(Z,\Gamma)p^{1-s}_{z,\gamma}(Z,\Gamma-h)\mathrm{d}\Gamma \\ &\quad - \int_{-\infty}^{+\infty} \mathrm{d}Z \int_{-\infty}^{+\infty} hp^s_{z,\gamma}(Z,\Gamma+h)p^{1-s}_{z,\gamma}(Z,\Gamma)\mathrm{d}\Gamma \\ &= -\int_{-\infty}^{+\infty} \mathrm{d}Z \int_{-\infty}^{+\infty} \sqrt{p_{z|\gamma}(Z \mid \Gamma+1)p_{z|\gamma}(Z \mid \Gamma)} \\ &\quad \times \sqrt{p_\gamma(\Gamma+1)p_\gamma(\Gamma)}\mathrm{d}\Gamma. \end{aligned} \tag{34}$$

But, according to (33)

$$\begin{aligned} p_\gamma(\Gamma+1)p_\gamma(\Gamma) &= (1-p)^2\delta(\Gamma+1)\delta(\Gamma) + p(1-p)\delta(\Gamma)^2 \\ &\quad + p(1-p)\delta(\Gamma+1)\delta(\Gamma-1) + p^2\delta(\Gamma)\delta(\Gamma-1). \end{aligned} \tag{35}$$

The last expression is zero for all Γ except $\Gamma = 0$. Therefore,

$$E[\gamma\psi(z,\gamma)] = -\sqrt{p(1-p)}\int_{-\infty}^{+\infty}\sqrt{p_{z|\gamma}(Z \mid 1)p_{z|\gamma}(Z \mid 0)}\mathrm{d}Z. \tag{36}$$

Similarly

$$\begin{aligned} &E\left[\psi^2(z,\gamma)\right] \\ &= \int_{-\infty}^{+\infty}\int_{\mathcal{G}'}\left(\sqrt{\frac{p_{z,\gamma}(Z,\Gamma+1)}{p_{z,\gamma}(Z,\Gamma)}} - \sqrt{\frac{p_{z,\gamma}(Z,\Gamma-1)}{p_{z,\gamma}(Z,\Gamma)}}\right)^2 \\ &\quad \times p_{z,\gamma}(Z,\Gamma)\mathrm{d}Z\mathrm{d}\Gamma \\ &= \int_{-\infty}^{+\infty}\int_{\mathcal{G}'}\Big(p_{z,\gamma}(Z,\Gamma+1) + p_{z,\gamma}(Z,\Gamma-1) \\ &\quad - 2\sqrt{p_{z,\gamma}(Z,\Gamma+1)p_{z,\gamma}(Z,\Gamma-1)}\Big)\mathrm{d}\Gamma\mathrm{d}Z. \end{aligned} \tag{37}$$

Using the following relations:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{\mathcal{G}'} p_{z,\gamma}(Z,\Gamma+1)\mathrm{d}\Gamma\mathrm{d}Z \\ &= \int_{-\infty}^{\infty}\int_{\mathcal{G}'} p_{z|\gamma}(Z \mid \Gamma+1)\Big((1-p)\delta(\Gamma+1) + p\delta(\Gamma)\Big)\mathrm{d}\Gamma\mathrm{d}Z = p \end{aligned} \tag{38}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{\mathcal{G}'} p_{z,\gamma}(Z,\Gamma-1)\mathrm{d}\Gamma\mathrm{d}Z \\ &= \int_{-\infty}^{\infty}\int_{\mathcal{G}'} p_{z|\gamma}(Z \mid \Gamma-1)\Big((1-p)\delta(\Gamma-1) + p\delta(\Gamma-2)\Big)\mathrm{d}\Gamma\mathrm{d}Z = 1-p \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{\mathcal{G}'}\sqrt{p_{z,\gamma}(Z,\Gamma+1)p_{z,\gamma}(Z,\Gamma-1)}\mathrm{d}\Gamma\mathrm{d}Z \\ &\qquad = \int_{-\infty}^{\infty}\int_{\mathcal{G}'}\sqrt{p_{z|\gamma}(Z \mid \Gamma+1)p_{z|\gamma}(Z \mid \Gamma-1)} \\ &\qquad\quad \times \sqrt{(1-p)\delta(\Gamma+1) + p\delta(\Gamma)} \\ &\qquad\quad \times \sqrt{(1-p)\delta(\Gamma-1) + p\delta(\Gamma-2)}\mathrm{d}\Gamma\mathrm{d}Z = 0 \end{aligned} \tag{40}$$

yields

$$E\left[\psi^2(z,\gamma)\right] = 1. \tag{41}$$

Substituting (36) and (41) into the Weiss-Weinstein bound (29) gives (12).

IV. Estimation Error Lower Bound

First, the new bound is derived for the special case of a white interruption process in the following lemma.

Lemma 4.1: Assume that in the system defined in Section II the sequences $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ are white in the sense that the pairs $(\gamma_k^{(i)}, \gamma_l^{(i)})$ are mutually independent for all $k \neq l$ (notice that in this case $P_{1j}^{(i)}(k \mid k-1) = p_k^{(i)}$ where the left-hand side (LHS) probability is defined in (4b) and the RHS probability is defined in (9)). Then, a lower bound on the estimation error variance of each interruption variable $\gamma_k^{(i)}$ is given by

$$E\left[\left(\hat{\gamma}_{k|k}^{(i)} - \gamma_k^{(i)}\right)^2\right] \geqslant p_k^{(i)}\left(1 - p_k^{(i)}\right) L_R^{(i)} L_H^{(i)} \tag{42}$$

where

$$L_R^{(i)} \triangleq \frac{\sqrt{\det\left[R_k^{(i)}(1) R_k^{(i)}(0)\right]}}{\det\left[\frac{1}{2}\left(R_k^{(i)}(1) + R_k^{(i)}(0)\right)\right]} \tag{43a}$$

and

$$L_H^{(i)} \triangleq \sqrt{\frac{\det\left[\Sigma_k^{-1} + \Delta H^{(i)T}\left(R_k^{(i)}(1) + R_k^{(i)}(0)\right)^{-1} \Delta H^{(i)}\right]^{-1}}{\det \Sigma_k}}. \tag{43b}$$

Proof: Assume, first, that the state vector x_k is known at each time instant k. Then, recalling (5) and the fact that $\{\gamma_k^{(i)}\}_{i=1}^N$ are independent of x_k and $\mathcal{Y}_{k-1}$ by the lemma's assumption yields

$$\begin{aligned} p_{\mathcal{Y}_k|\gamma_k,x_k}&(\Upsilon_k \mid \Gamma_k, X_k) \\ &= p_{y_k|\gamma_k,x_k,\mathcal{Y}_{k-1}}(Y_k \mid \Gamma_k, X_k, \Upsilon_{k-1}) \\ &\quad \times p_{\mathcal{Y}_{k-1}|\gamma_k,x_k}(\Upsilon_{k-1} \mid \Gamma_k, X_k) \\ &= \left(\prod_{i=1}^N p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid \Gamma_k^{(i)}, X_k)\right) p_{\mathcal{Y}_{k-1}|x_k}(\Upsilon_{k-1} \mid X_k) \end{aligned} \tag{44}$$

where

$$\Gamma_k \triangleq \left[\Gamma_k^{(1)}, \dots, \Gamma_k^{(N)}\right]. \tag{45}$$

Therefore,

$$\begin{aligned} p_{\mathcal{Y}_k|\gamma_k^{(i)},x_k}&(\Upsilon_k \mid \Gamma_k^{(i)}, X_k) \\ &= p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid \Gamma_k^{(i)}, X_k) \\ &\quad \times \left(\prod_{j=1, j\neq i}^N p_{y_k^{(j)}|x_k}(Y_k^{(j)} \mid X_k)\right) p_{\mathcal{Y}_{k-1}|x_k}(\Upsilon_{k-1} \mid X_k). \end{aligned} \tag{46}$$

Now, according to Lemma 3.1, the lower bound on the estimation error variance of each interruption variable $\gamma_k^{(i)}$ is as shown in (47) at the bottom of the page. By definition

$$y_k^{(i)} \mid \gamma_k^{(i)}, x_k \sim \mathcal{N}\left(\left(H^{(i)} + \gamma_k^{(i)} \Delta H^{(i)}\right) x_k, R_k^{(i)}(\gamma_k^{(i)})\right). \tag{48}$$

Therefore, the integral on the RHS of (47) is equal to

$$\begin{aligned} \int_{-\infty}^{+\infty} &\sqrt{p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid 0, x_k) p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid 1, x_k)} \mathrm{d}Y_k^{(i)} \\ &= \frac{1}{(2\pi)^{m_i/2}\left(\det R_k^{(i)}(1) \det R_k^{(i)}(0)\right)^{1/4}} \\ &\quad \times \int_{-\infty}^{+\infty} \exp\left[\Lambda(Y_k^{(i)})\right] \mathrm{d}Y_k^{(i)} \end{aligned} \tag{49}$$

where

$$\begin{aligned} \Lambda(Y_k^{(i)}) \triangleq &-\frac{1}{4}\left\{\left[Y_k^{(i)} - H^{(i)} x_k\right]^T R_k^{(i)}(0)^{-1} \left[Y_k^{(i)} - H^{(i)} x_k\right]\right. \\ &+ \left[Y_k^{(i)} - H^{(i)} x_k - \Delta H^{(i)} x_k\right]^T R_k^{(i)}(1)^{-1} \\ &\left.\times \left[Y_k^{(i)} - H^{(i)} x_k - \Delta H^{(i)} x_k\right]\right\}. \end{aligned} \tag{50}$$

Now, using the following definitions:

$$\widetilde{Y} \triangleq Y_k^{(i)} - H^{(i)} x_k \tag{51a}$$

$$\alpha \triangleq \left[R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right]^{-1} R_k^{(i)}(1)^{-1} \Delta H^{(i)} x_k \tag{51b}$$

in (50) yields

$$\begin{aligned} \Lambda(Y_k^{(i)}) = &-\frac{1}{4}\left\{\widetilde{Y}^T\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\widetilde{Y}\right. \\ &-\widetilde{Y}^T\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\alpha \\ &-\alpha^T\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\widetilde{Y} \\ &+\alpha^T\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\alpha \\ &\left.+x_k^T \Delta H^{(i)T}\left(R_k^{(i)}(1) + R_k^{(i)}(0)\right)^{-1} \Delta H^{(i)} x_k\right\} \\ = &-\frac{1}{4}\left\{\left[\widetilde{Y} - \alpha\right]^T\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\left[\widetilde{Y} - \alpha\right]\right. \\ &\left.+x_k^T \Delta H^{(i)T}\left(R_k^{(i)}(1) + R_k^{(i)}(0)\right)^{-1} \Delta H^{(i)} x_k\right\} \\ = &-\frac{1}{2}\left[Y_k^{(i)} - H^{(i)} x_k - \alpha\right]^T\left[\frac{1}{2}\left(R_k^{(i)}(0)^{-1} + R_k^{(i)}(1)^{-1}\right)\right] \\ &\times\left[Y_k^{(i)} - H^{(i)} x_k - \alpha\right] \\ &-\frac{1}{4} x_k^T \Delta H^{(i)T}\left(R_k^{(i)}(1) + R_k^{(i)}(0)\right)^{-1} \Delta H^{(i)} x_k. \end{aligned} \tag{52}$$

$$\begin{aligned} E\left[\left(\hat{\gamma}_{k|k}^{(i)} - \gamma_k^{(i)}\right)^2 \mid x_k\right] &\geqslant p_k^{(i)}\left(1 - p_k^{(i)}\right)\left[\int_{-\infty}^{+\infty}\sqrt{p_{\mathcal{Y}_k|\gamma_k^{(i)},x_k}(\Upsilon_k \mid 0, x_k) p_{\mathcal{Y}_k|\gamma_k^{(i)},x_k}(\Upsilon_k \mid 1, x_k)} \mathrm{d}\Upsilon_k\right]^2 \\ &= p_k^{(i)}(1 - p_k^{(i)})\left[\int_{-\infty}^{+\infty}\sqrt{p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid 0, x_k) p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid 1, x_k)}\right. \\ &\quad \left.\times\left(\prod_{j=1,j\neq i}^N p_{y_k^{(j)}|x_k}(Y_k^{(j)} \mid X_k)\right)\left(\int_{-\infty}^{+\infty} p_{\mathcal{Y}_{k-1}|x_k}(\Upsilon_{k-1} \mid x_k) \mathrm{d}\Upsilon_{k-1}\right) \mathrm{d}Y_k^{(1)} \cdots \mathrm{d}Y_k^{(N)}\right]^2 \\ &= p_k^{(i)}(1 - p_k^{(i)})\left[\int_{-\infty}^{+\infty}\sqrt{p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)} \mid 0, x_k) p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k \mid 1, x_k)} \mathrm{d}Y_k^{(i)}\right]^2. \end{aligned} \tag{47}$$

Substituting (52) into (49) yields

$$\begin{aligned}&\int_{-\infty}^{+\infty}\sqrt{p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)}\mid 0,x_k)p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)}\mid 1,x_k)\mathrm{d}Y_k^{(i)}}\\ &=\sqrt{\frac{\sqrt{\det\left[R_k^{(i)}(1)^{-1}R_k^{(i)}(0)^{-1}\right]}}{\det\left[\frac{1}{2}\left(R_k^{(i)}(1)^{-1}+R_k^{(i)}(0)^{-1}\right)\right]}}\\ &\quad\times\exp\left[-\frac{1}{4}x_k^T\Delta H^{(i)^T}\left(R_k^{(i)}(1)+R_k^{(i)}(0)\right)^{-1}\Delta H^{(i)}x_k\right].\end{aligned}\tag{53}$$

But

$$\begin{aligned}&\frac{\sqrt{\det\left[R_k^{(i)}(1)^{-1}R_k^{(i)}(0)^{-1}\right]}}{\det\left[\frac{1}{2}\left(R_k^{(i)}(1)^{-1}+R_k^{(i)}(0)^{-1}\right)\right]}\\ &=\frac{\sqrt{\det\left[R_k^{(i)}(1)^{-1}R_k^{(i)}(0)^{-1}\right]}}{\det R_k^{(i)}(1)^{-1}\det\left[\frac{1}{2}\left(R_k^{(i)}(1)+R_k^{(i)}(0)\right)\right]\det R_k^{(i)}(0)^{-1}}\\ &=L_R^{(i)}\end{aligned}\tag{54}$$

where $L_R^{(i)}$ is defined in (43a). Therefore,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\sqrt{p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)}\mid 0,x_k)p_{y_k^{(i)}|\gamma_k^{(i)},x_k}(Y_k^{(i)}\mid 1,x_k)\mathrm{d}Y_k^{(i)}}\\ &=\sqrt{L_R^{(i)}}\exp\left[-\frac{1}{4}x_k^T\Delta H^{(i)^T}\left(R_k^{(i)}(1)+R_k^{(i)}(0)\right)^{-1}\Delta H^{(i)}x_k\right].\end{aligned}\tag{55}$$

Finally, substituting the last result into (47), taking the mathematical expectation of both sides of the expression, and using the fact that $x_k\sim\mathcal{N}\left(0,\Sigma_k\right)$ yields

$$\begin{aligned}&E\left[\left(\hat{\gamma}_{k|k}^{(i)}-\gamma_k^{(i)}\right)^2\right]\\ &\geqslant p_k^{(i)}\left(1-p_k^{(i)}\right)L_R^{(i)}\frac{1}{\sqrt{(2\pi)^n\det(\Sigma_k)}}\\ &\quad\times\int_{-\infty}^{+\infty}\exp\Bigg[-\frac{1}{2}X_k^T\Delta H^{(i)^T}\left(R_k^{(i)}(1)+R_k^{(i)}(0)\right)^{-1}\Delta H^{(i)}X_k\\ &\quad-\frac{1}{2}X_k^T\Sigma_k^{-1}X_k\Bigg]\mathrm{d}X_k\\ &=p_k^{(i)}\left(1-p_k^{(i)}\right)L_R^{(i)}L_H^{(i)}\end{aligned}\tag{56}$$

where $L_H^{(i)}$ is defined in (43b). □

The main result of this correspondence is now stated in the following theorem.

Theorem 4.1: A lower bound on the estimation error variance of each interruption variable in the system defined in Section II is given by

$$E\left[\left(\hat{\gamma}_{k|k}^{(i)}-\gamma_k^{(i)}\right)^2\right]\geqslant\varkappa_P^{(i)}L_R^{(i)}L_H^{(i)}\tag{57}$$

where $L_R^{(i)}$ and $L_H^{(i)}$ are given in (43a) and (43b), respectively, and $\varkappa_P^{(i)}$ is defined as

$$\varkappa_P^{(i)}\triangleq\left[p_k^{(i)}\left(1-p_k^{(i)}\right)-p_{k-1}^{(i)}\left(1-p_{k-1}^{(i)}\right)\left(P_{11}^{(i)}-P_{10}^{(i)}\right)^2\right].\tag{58}$$

Proof: Given the value of γ_{k-1} the system reduces, due to the Markov property, to the case treated by Lemma 4.1. Therefore,

$$E\left[\left(\hat{\gamma}_{k|k}^{(i)}-\gamma_k^{(i)}\right)^2\mid\gamma_{k-1}^{(i)}=j\right]\geqslant P_{1j}^{(i)}\left(1-P_{1j}^{(i)}\right)L_R^{(i)}L_H^{(i)}.\tag{59}$$

Taking mathematical expectation of both sides of (59) and using the smoothing property of the conditional expectation yields

$$\begin{aligned}&E\left[\left(\hat{\gamma}_{k|k}^{(i)}-\gamma_k^{(i)}\right)^2\right]\\ &\geqslant\left[P_{10}^{(i)}\left(1-P_{10}^{(i)}\right)\left(1-p_{k-1}^{(i)}\right)+P_{11}^{(i)}\left(1-P_{11}^{(i)}\right)p_{k-1}^{(i)}\right]\\ &\quad\times L_R^{(i)}L_H^{(i)}\end{aligned}\tag{60}$$

which gives (57), (58) after rearranging terms. □

Remark 4.1: In the case of a homogeneous chain in steady state ($p_k^{(i)}=p^{(i)}=$constant) the term $\varkappa_P^{(i)}$ in the lower bound takes the form

$$\varkappa_P^{(i)}=p^{(i)}\left(1-p^{(i)}\right)\left[1-\left(P_{11}^{(i)}-P_{10}^{(i)}\right)^2\right].\tag{61}$$

A. Lower Bound Computation Algorithm

The lower bound presented in Theorem 4.1 can be easily evaluated using the following recursive algorithm.

1) The state vector covariance matrix is initialized with Σ_0.
2) At each time step $k=1,2,\dots,$ the state vector covariance matrix is updated via the recursion

$$\Sigma_k=\Phi\Sigma_{k-1}\Phi^T+GQG^T.\tag{62}$$

The *a priori* probabilities $p_k^{(i)}$ are updated via the recursion

$$p_k^{(i)}=P_{11}^{(i)}p_{k-1}^{(i)}+P_{10}^{(i)}\left(1-p_{k-1}^{(i)}\right).\tag{63}$$

3) Finally, (57) together with (43a), (43b), and (58) is used to compute the lower bound on the estimation error variance of each interruption variable at time k.

Note that the lower bounds for different interruption variables are independent and can be evaluated separately.

V. Discussion

In this section, the effects of various factors on the new lower bound and its subsequent properties are presented and discussed.

A. Effect of Interruption Chain Transitional Determinism

It follows from (59) and (60) that

$$\varkappa_P^{(i)}=E\left[P_{1\gamma_{k-1}^{(i)}}^{(i)}\left(1-P_{1\gamma_{k-1}^{(i)}}^{(i)}\right)\right].\tag{64}$$

This term is close to zero if the transition probabilities $P_{1j}^{(i)}$ are close to either 1 or 0, or, in other words, if the interruption chain $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ is almost transitionally deterministic. The higher the transition uncertainty of the chain, the farther are the transition probabilities from 1 and 0 and the larger is $\varkappa_P^{(i)}$. It can be concluded, therefore, that the term $\varkappa_P^{(i)}$ can be used to define a measure of the interruption chain transitional determinism, and expresses the effect of that determinism on the proposed bound: in an almost transitionally deterministic chain the lower bound is close to zero, as could be expected.

B. Effect of Measurement Noise Sensitivity to Interruption Process

According to Theorem 4.1, the effect of the sensitivity of the measurement noise in channel i to switchings of the interruption process

$\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ is expressed by the term $L_R^{(i)}$. This term has the following property.

Proposition 5.1: For any positive definite $R_k^{(i)}(1)$ and $R_k^{(i)}(0)$

$$L_R^{(i)} \leqslant 1 \tag{65}$$

where the equality exists iff

$$R_k^{(i)}(1) = R_k^{(i)}(0) \tag{66}$$

i.e., when $R_k^{(i)}$ is not a function of $\gamma_k^{(i)}$.

Proof: First, $R_k^{(i)}(0)$, being a positive-definite matrix, can be written in the following form:

$$R_k^{(i)}(0) = SS^T, \qquad \det S \neq 0. \tag{67}$$

Hence,

$$\begin{aligned} L_R^{(i)} &= \frac{\sqrt{\det\left(R_k^{(i)}(1)SS^T\right)}}{\det\left(\frac{1}{2}\left(R_k^{(i)}(1)+SS^T\right)\right)} \\ &= \frac{\sqrt{\det\left(R_k^{(i)}(1)SS^T\right)}}{\det(S)\det(S^T)\det\left(\frac{1}{2}\left(S^{-1}R_k^{(i)}(1)S^{-T}+I\right)\right)} \\ &= \frac{\sqrt{\det\left(S^{-1}R_k^{(i)}(1)S^{-T}\right)}}{\det\left(\frac{1}{2}\left(S^{-1}R_k^{(i)}(1)S^{-T}+I\right)\right)}. \end{aligned} \tag{68}$$

Let $\{\lambda_j\}_{j=1}^{m_i}$ be the eigenvalues of the positive definite matrix $S^{-1}R_k^{(i)}(1)S^{-T}$. Then the eigenvalues of the matrix

$$\frac{1}{2}\left(S^{-1}R_k^{(i)}(1)S^{-T}+I\right)$$

are equal to $\{1/2(\lambda_j+1)\}_{j=1}^{m_i}$. Therefore, using (68)

$$L_R^{(i)} = \prod_{j=1}^{m_i}\frac{2\sqrt{\lambda_j}}{\lambda_j+1} = \prod_{j=1}^{m_i}\left(1-\frac{(1-\sqrt{\lambda_j})^2}{\lambda_j+1}\right) \leqslant 1 \tag{69}$$

which is (65). Moreover, equality in (69) takes place iff

$$\lambda_j = 1, \qquad j = 1, 2, \ldots, m_i \tag{70}$$

which is equivalent to requiring that

$$S^{-1}R_k^{(i)}(1)S^{-T} = I \tag{71}$$

yielding (66). □

It follows from Proposition 5.1 that the farther are the measurement noise covariance matrices $R_k^{(i)}(1)$ and $R_k^{(i)}(0)$ from one another, the lower is the bound. The intuitive reason for this phenomenon is that the larger the changes in the measurement noise covariance matrix due to switchings in the interruption process $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$, the easier is its estimation and the smaller is its attainable estimation error variance.

C. Effect of State Estimability

Examining the term $L_H^{(i)}$ given in (43b) one can see that it is a square root of a ratio of determinants of two covariance matrices. The numerator of the ratio is the determinant of the state estimation covariance matrix, obtained after a single measurement update has been performed at time k using a Kalman filter with an effective measurement information contribution equal to

$$\Delta H^{(i)T}\left(R_k^{(i)}(1)+R_k^{(i)}(0)\right)^{-1}\Delta H^{(i)}.$$

The denominator is simply the determinant of the state covariance at time k. It is well known from the theory of Gaussian vectors that for $z \sim \mathcal{N}(\hat{z}, P_z)$ the value of $\sqrt{\det P_z}$ is proportional to the volume of the uncertainty ellipsoid, defined as

$$Z^T P_z^{-1} Z \leqslant 1 \tag{72}$$

where Z is a realization of z. Therefore, the term $L_H^{(i)}$ that satisfies

$$L_H^{(i)} \leqslant 1 \tag{73}$$

is equal to the ratio of the volumes of the uncertainty ellipsoids, after and before a single measurement update with the aforementioned effective measurement information contribution, respectively. This uncertainty ellipsoid volume ratio can be used to define a measure of state estimability. The notion of estimability as a binary measure of the ability to reduce the state uncertainty using a linear filter was originally proposed by Baram and Kailath in [14]. Expanding upon this notion, it can be stated that the inverse of the term $L_H^{(i)}$ can be used as a measure for the system's state estimability, and expresses the effect of that estimability on the proposed lower bound.

D. Lower Bound Tightness

Upon examining the effects of the various factors on the proposed lower bound as discussed earlier, it becomes clear that its tightness depends heavily on the whiteness of the interruption process. Thus, according to Lemma 4.1, the lower bound in the white interruption sequence case is equal to the interruption variable variance multiplied by the terms $L_R^{(i)}$ and $L_H^{(i)}$. Therefore, if the system measurements in channel i are insensitive to the interruption process, in which case both $L_R^{(i)}$ and $L_H^{(i)}$ are equal to 1, the lower bound is equal to the interruption process variance. In this case, there exists an efficient estimator $\hat{\gamma}_{k|k}^{(i)} = E[\gamma_k^{(i)}] = p_k^{(i)}$. In practice, this may happen if $R^{(i)}(0) \approx R^{(i)}(1)$ and the system's state estimability measure is small. If, on the other hand, either $L_R^{(i)}$ or $L_H^{(i)}$ is close to zero, which may happen if either $R^{(i)}(0)$ and $R^{(i)}(1)$ are very different from one another or the system's state estimability measure is large, the lower bound is close to zero, which expresses the feasibility of perfectly estimating the interruption indicators. In fact, when $\tilde{H}_k^{(i)}(\gamma_k^{(i)}) = 0$ for some realization of $\gamma_k^{(i)}$ and $R^{(i)}(0) \approx R^{(i)}(1) \approx 0$, such an estimate can be provided by a simple χ^2-based hypothesis test.

It should be noted that the aforementioned two cases, where efficient estimators exist, correspond to the conditions (23b) and (23a), respectively, of Corollary 3.1. Moreover, according to that corollary, these are the only cases where the lower bound is tight.

In the case of a colored interruption process

$$\begin{aligned} \varkappa_P^{(i)} &= p_k^{(i)}\left(1-p_k^{(i)}\right) - p_{k-1}^{(i)}\left(1-p_{k-1}^{(i)}\right)\left(P_{11}^{(i)}-P_{10}^{(i)}\right)^2 \\ &\leqslant p_k^{(i)}\left(1-p_k^{(i)}\right) \end{aligned} \tag{74}$$

so that even if the system measurements in channel i are insensitive to $\gamma_k^{(i)}$, the lower bound is still smaller than the interruption process variance. In the case of a highly correlated Markov chain one has

$$\left(P_{11}^{(i)}-P_{10}^{(i)}\right)^2 \approx 1 \tag{75}$$

TABLE I
TEST CASES OF THE LOWER BOUND NUMERICAL STUDY

Description	Q	φ	P_{11}	P_{10}	$R(1)$	$R(0)$
Nominal	0.25	0.9987	0.7	0.3	9	9
Slow interruption process	0.25	0.9987	0.95	0.05	9	9
High process noise	36	0.8	0.7	0.3	9	9
Large difference between $R(1)$ and $R(0)$	0.25	0.9987	0.7	0.3	17	1
High measurement noise	0.25	0.9987	0.5	0.5	100	100
Low measurement noise	0.25	0.9987	0.5	0.5	0.01	0.01

which renders the bound approximately zero even if $L_R^{(i)} L_H^{(i)} \approx 1$. The reason for this phenomenon lies in the conditioning on γ_{k-1} in the derivation procedure. Because of this conditioning, the lower bound given in Theorem 4.1 must be valid also for estimators that explicitly use the value of γ_{k-1}. In systems with highly correlated interruption processes

$$\Pr\left\{\gamma_k^{(i)} = \gamma_{k-1}^{(i)}\right\} \approx 1 \tag{76a}$$

or

$$\Pr\left\{\gamma_k^{(i)} = \gamma_{k-1}^{(i)}\right\} \approx 0 \tag{76b}$$

driving the *a priori* uncertainty (and, hence, the estimation error lower bound) of the interruption variables to zero.

Another factor affecting the lower bound tightness is the state vector estimation accuracy. The proof of Lemma 4.1 assumes that the state vector x_k is known. This assumption implies that the lower bound given in Lemma 4.1 and, therefore, in Theorem 4.1, must be valid also for estimators that explicitly use the value of x_k. In other words, the proposed lower bound does not consider the impact of the state vector estimation error on the interruption variable estimation accuracy. It follows, therefore, that the lower bound becomes less tight—and, hence, its utility decreases—as, e.g., the process noise increases.

VI. NUMERICAL EXAMPLE

A numerical simulation study has been carried out to demonstrate the usage and properties of the new lower bound. In this study, the lower bound is used to examine the efficiency of the interacting multiple model (IMM) estimation algorithm [6]. The IMM algorithm is a powerful tool in filtering of hybrid systems. The estimator consists of a bank of Kalman filters designed each for a different discrete mode of the system. The residuals produced by these filters are used to form the mode likelihood functions, which then serve in a hypothesis-testing mechanism. The stage that makes the IMM algorithm particularly powerful is the interaction stage. During this stage, the estimates of different Kalman filters are mixed using mixing probabilities such that the less likely estimates are "punished" in the sense that their covariance matrices grow relative to those of more likely modes.

As an illustrative example, the following simple scalar system is considered. Let $\{x_k\}_{k=0}^{\infty}$ be a scalar stationary zero-mean Gaussian process with the autocorrelation function

$$R_x(k,l) = 100\varphi^{|k-l|}, \qquad 0 < \varphi < 1. \tag{77}$$

Such a process can be generated, e.g., by the following first-order dynamics:

$$x_{k+1} = \varphi x_k + w_{k+1} \tag{78}$$

where $x_0 \sim \mathcal{N}(0,100)$ and the process noise $\{w_k\}_{k=1}^{\infty}$ is a white sequence with $w_k \sim \mathcal{N}(0,Q)$, such that

$$Q = 100\left(1-\varphi^2\right). \tag{79}$$

The observation process in this example is also scalar and is defined as

$$y_{k+1} = (1-\gamma_{k+1})\, x_{k+1} + v_{k+1} \tag{80}$$

where the measurement noise $\{v_k\}_{k=1}^{\infty}$ is a white sequence independent of the process noise $\{w_k\}_{k=1}^{\infty}$, and the initial state x_0, with $v_k \sim \mathcal{N}(0, R(\gamma_k))$. The interruption process $\{\gamma_k\}_{k=0}^{\infty}$ is a Bernoulli Markov chain with

$$\Pr\{\gamma_0 = 1\} = 0 \tag{81a}$$

$$\Pr\{\gamma_k = 1 \mid \gamma_{k-1} = j\} = P_{1j}, \qquad j \in \{0,1\}. \tag{81b}$$

In this example Σ_k is constant ($\Sigma_k = 100$), $N = 1$, $H^{(1)} = 1$, and $\Delta H^{(1)} = -1$. Therefore, the lower bound given by Theorem 4.1 takes the form

$$E\left[\left(\hat{\gamma}_{k|k} - \gamma_k\right)^2\right] \geqslant \left[p_k\,(1-p_k) - p_{k-1}\,(1-p_{k-1})\,(P_{11}-P_{10})^2\right] \times \frac{2\sqrt{R(1)R(0)}}{R(1)+R(0)} \frac{1}{\sqrt{1+\frac{100}{R(1)+R(0)}}} \tag{82}$$

where the probabilities p_k can be computed using the following recursion:

$$p_k = P_{11}p_{k-1} + P_{10}\,(1-p_{k-1}) \tag{83}$$

and $p_0 = 0$.

Concerning the application of the IMM algorithm to the system defined above, note that it has only two modes corresponding to $\gamma_k = 1$ and $\gamma_k = 0$. Therefore, the IMM algorithm comprises the following steps. Note that at each time step $k+1$ the quantities: $\hat{x}_{k|k,k}^{(j)}$, $\widehat{P}_{k|k,k}^{(j)}$ for $j \in \{0,1\}$ and $\hat{\gamma}_{k|k}$ are available from previous calculations.

1) Mode time propagation

$$\hat{\gamma}_{k+1|k} = P_{11}\hat{\gamma}_{k|k} + P_{10}\left(1-\hat{\gamma}_{k|k}\right). \tag{84}$$

2) Mixing ($j \in \{0,1\}$)

$$\hat{x}_{k|k+1,k}^{(j)} = \mu_j\,\hat{x}_{k|k,k}^{(1)} + (1-\mu_j)\,\hat{x}_{k|k,k}^{(0)} \tag{85a}$$

$$\widehat{P}_{k|k+1,k}^{(j)} = \mu_j\left[\widehat{P}_{k|k,k}^{(1)} + \left(\hat{x}_{k|k,k}^{(1)} - \hat{x}_{k|k+1,k}^{(j)}\right)^2\right] + (1-\mu_j)\left[\widehat{P}_{k|k,k}^{(0)} + \left(\hat{x}_{k|k,k}^{(0)} - \hat{x}_{k|k+1,k}^{(j)}\right)^2\right] \tag{85b}$$

where

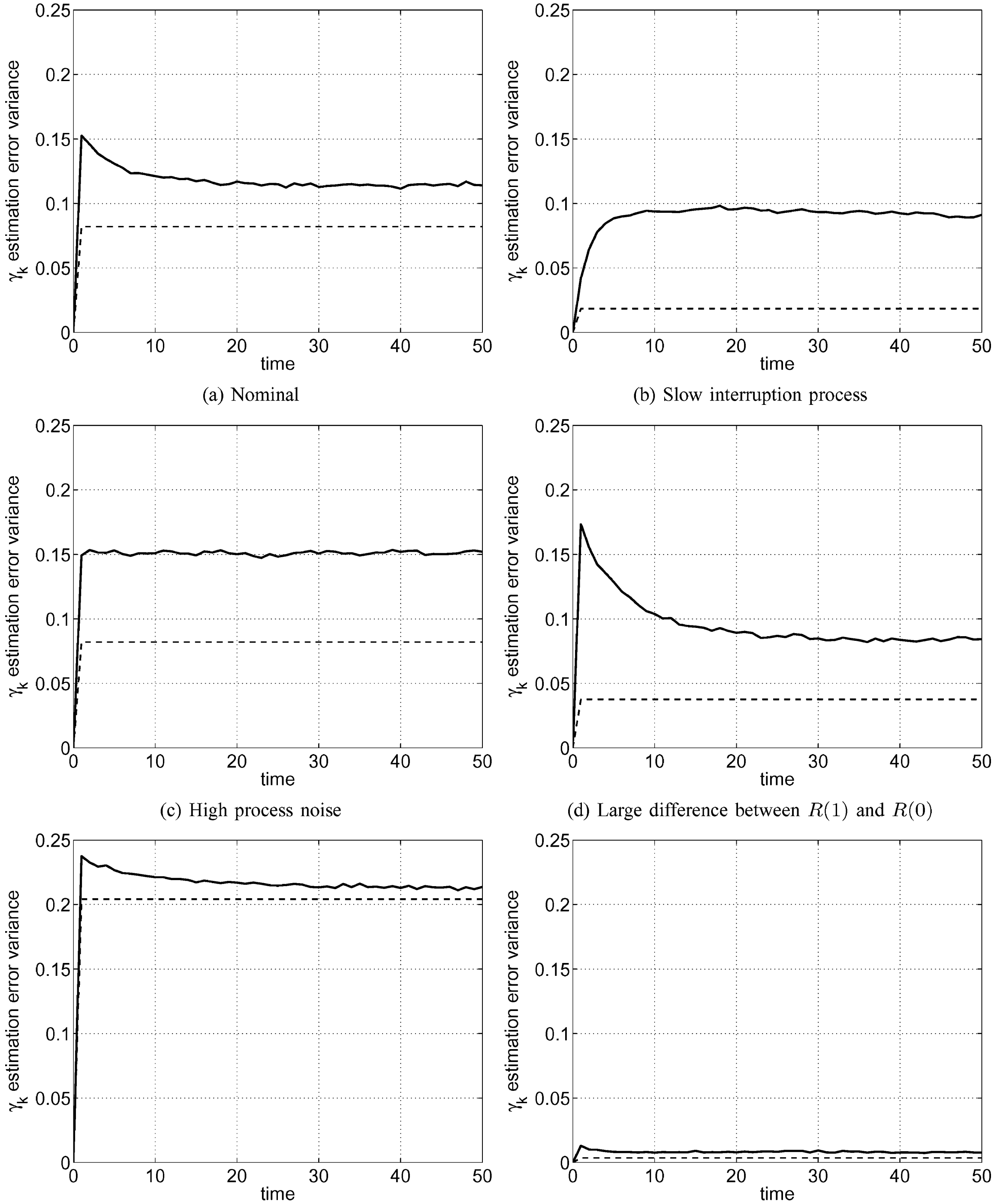
$$\mu_1 = \frac{P_{11}\hat{\gamma}_{k|k}}{\hat{\gamma}_{k+1|k}} \tag{86a}$$

$$\mu_0 = \frac{(1-P_{11})\,\hat{\gamma}_{k|k}}{1-\hat{\gamma}_{k+1|k}}. \tag{86b}$$

3) State time propagation ($j \in \{0,1\}$)

$$\hat{x}_{k+1|k+1,k}^{(j)} = \varphi\hat{x}_{k|k+1,k}^{(j)} \tag{87a}$$

$$\widehat{P}_{k+1|k+1,k}^{(j)} = \varphi^2\widehat{P}_{k|k+1,k}^{(j)} + Q. \tag{87b}$$



(a) Nominal

(b) Slow interruption process

(c) High process noise

(d) Large difference between $R(1)$ and $R(0)$

(e) High measurement noise

(f) Low measurement noise

Fig. 1. γ_k estimation error variance achieved by IMM (bold solid line) versus the lower bound (thin dashed line).

4) Mode measurement update

$$\hat{\gamma}_{k+1|k+1} = \frac{f_1 \hat{\gamma}_{k+1|k}}{f_1 \hat{\gamma}_{k+1|k} + f_0 \left(1 - \hat{\gamma}_{k+1|k}\right)} \tag{88}$$

where

$$f_j = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2\sigma_j^2}\left(y_{k+1} - (1-j)\hat{x}^{(j)}_{k+1|k+1,k}\right)^2\right], \quad j \in \{0,1\} \tag{89}$$

and

$$\sigma_j^2 = (1-j)^2 \widehat{P}^{(j)}_{k+1|k+1,k} + R(j), \qquad j \in \{0,1\}. \tag{90}$$

5) State measurement update ($j \in \{0,1\}$)

$$K_j = \frac{\widehat{P}^{(j)}_{k+1|k+1,k}(1-j)}{\sigma_j^2} \tag{91a}$$

$$\hat{x}^{(j)}_{k+1|k+1,k+1} = \hat{x}^{(j)}_{k+1|k+1,k} + K_j\left[y_{k+1} - (1-j)\hat{x}^{(j)}_{k+1|k+1,k}\right] \tag{91b}$$

$$\widehat{P}^{(j)}_{k+1|k+1,k+1} = \left[1 - K_j(1-j)\right] \widehat{P}^{(j)}_{k+1|k+1,k} \tag{91c}$$

where σ_j is given by (90).

Applying these steps the estimate of γ_k at each time step k is given by $\hat{\gamma}_{k|k}$.

Six test cases, whose numerical parameters are summarized in Table I, were studied. 20 000 Monte Carlo runs were used to compute the estimation error variance of the IMM algorithm in each case.

The results of the study are presented in Fig. 1. One can see that in the nominal case [Fig. 1(a)], where the process noise is relatively small, the IMM estimation error variance and the lower bound are quite close to each other: the lower bound is about 72% of the IMM estimation error variance. The immediate conclusion is that in the nominal case the IMM algorithm is close to being optimal in the mean-square sense. In the case of the slow interruption process [Fig. 1(b)], both the IMM estimation error variance and the lower bound become smaller, but in this case their difference grows: the lower bound is now only 21% of the IMM estimation error variance [see Fig. 1(b)]. This result agrees with the claim that the lower bound given by Theorem 4.1 is no longer tight in the case of slow (i.e., highly correlated) Markov chains. In the case of high process noise [Fig. 1(c)] the IMM estimation error variance grows relative to the nominal case while the lower bound remains the same, such that it constitutes 54% of the IMM estimation error variance. This result corresponds to the fact that the lower bound does not take into account the effect of state vector estimation error on the estimation accuracy of the interruption variables. The effect of different measurement noise variances in $\gamma_k = 1$ and $\gamma_k = 0$ is shown in Fig. 1(d). Both the IMM estimation error variance and the lower bound become smaller than in the nominal case. The lower bound in this case is about 45% of the IMM estimation error variance, which may mean that either the lower bound is not tight or the IMM algorithm is not optimal. The last two cases [Fig. 1(e) and (f)] correspond to the two situations where efficient estimators may exist. In these cases, the interruption sequence is white, the measurement noise is unaffected by the interruption sequence, and the system's state estimability measure is either very small [Fig. 1(e)] or very large [Fig. 1(f)]. One can see that the IMM estimation error variances are very close to the lower bounds, which means that the IMM algorithm in these cases is efficient and is, therefore, optimal in the mean-square sense.

VII. Conclusion

A lower bound on the estimation error of interruption variables in systems with uncertain measurements has been presented. Contrary to the recently proposed CRLB-type lower bound for this type of systems, the lower bound presented here, which is derivable using the well-known Weiss–Weinstein lower bound, is nontrivial. It can be easily evaluated independently for each measurement channel using a simple recursive algorithm.

The new lower bound depends on a measure of the interruption chain transitional determinism, the measurement noise sensitivity to interruption process switchings, and a measure of the system's state estimability. However, the effect of the state vector estimation error on the estimation accuracy of the interruption variables is not taken into account.

It is shown that the proposed lower bound can be tight in the case of white interruption processes and extremely high or extremely low state estimability. Its tightness reduces in the case of highly correlated interruption processes.

The use of the lower bound and some of its properties are illustrated via a simple numerical example. The proposed result can be implemented in practical applications to examine fault detectability in systems with independent fault-prone sensors. The lower bound is particularly useful in systems with low process noise (such as some spacecraft applications) and almost white Markovian interruption sequences.

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