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Brief Paper

Square Root Filtering via Covariance and Information Eigenfactors*

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Key Words—(Square root filtering); Kalman filters; filtering; state estimation; recursive algorithms; numerical methods; eigenvalues; decomposition; signal processing.

Abstract—Two new square root Kalman filtering algorithms are presented. Both algorithms are based on the spectral $V - \Lambda$ of the covariance matrix where V is the matrix whose columns are the eigenvectors of the covariance and Λ is the diagonal matrix of its eigenvalues. The algorithms use the covariance mode in the time propagation stage and the information mode in the measurement update stage. This switch between modes, which is trivial in the $V - \Lambda$ representation, increases the efficiency of the algorithms. In the first algorithm, which is a continuous/discrete one, the V and $\Lambda^{1/2}$ matrices are propagated in time in a continuous manner, while the measurement update is a discrete time procedure. In the second algorithm, which is a discrete/discrete one, the time propagation of the $V - \Lambda^{1/2}$ factors is performed in discrete time too, using a procedure which is similar to the one used for the discrete measurement update. The discrete propagation and the measurement update are based on singular value decomposition algorithms. The square root nature of the algorithms is demonstrated numerically through a typical example. While promising all the virtues of square root routines, the $V - \Lambda$ filters are also characterized by their ability to exhibit singularities as they occur.

1. *Introduction*

SOON AFTER the introduction of the Kalman filter (KF) (Kalman, 1960; Kalman and Bucy, 1961), it has been recognized that numerical problems may arise in its implementation in practice (Bellantoni and Dodge, 1967; Potter and Stern, 1963). Basically, the problems stem from the use of short word-length computers to compute the covariance matrix in ill-conditioned cases. This led to the computation of non-positive definite covariance matrices, which often also resulted in divergence. To overcome these problems, square root (SR) algorithms were developed. The most popular SR method is Bierman's discrete time U-D algorithm (Bierman, 1976; Thornton and Bierman, 1975). The algorithm uses a decomposition of the covariance matrix into a UDU^T form where U is a unit upper triangular matrix and D is a diagonal one. At each measurement instant, U and D are updated and propagated between measurements.

This paper introduces two new discrete-time square root $V - \Lambda$ algorithms; namely, a continuous/discrete (continuous time update/discrete measurement update) and a discrete/discrete algorithm. The continuous/discrete algorithm relies upon the results presented in Oshman and Bar–Itzhack (1985a), combining the continuous time update method developed there with a new, discrete, measurement update. In the discrete/discrete version a new discrete time update is presented. The new algorithms employ singular value decomposition (SVD), for which there exist today efficient and stable algorithms.

From a computational viewpoint, the new algorithms presented in this paper are more complex than other SR procedures that exist today. This is so because of the reliance upon the SVD technique (as opposed to more efficient orthogonal transformations, on which other SR algorithms are based). Nevertheless, the new algorithms may be of great importance in certain applications, e.g. where loss of accuracy due to harsh numerics is expected, or where continuous monitoring of the eigenfactors is necessary in order to reveal singularities as they occur and to identify those state subsets that are nearly dependent (Bierman, 1977, p. 100; Lawson and Hanson, 1974, p. 72). It is believed that as the SVD is becoming today a tool of primary importance in control theory, further research will eventually lead to the development of new SVD algorithms of higher efficiency, to the benefit of the new $V - \Lambda$ filters. Moreover, with the rapid emergence of very large scale integration (VLSI), new parallel computing structures have been introduced for efficient, real time implementation of matrix arithmetic algorithms such as Cholesky decomposition, eigenvalue decomposition etc. (Ahmed *et al.*, 1982). It is anticipated that the power of the new technology will be used to develop special purpose processor arrays that will perform the SVD faster and more cheaply than the conventional general purpose single processor computer.

2. *Discrete-time $V - \Lambda$ measurement update.*

In the ensuing, the following notation will be used to describe the measurement equation:

$$\mathbf{y}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \tag{2.1}$$

where $\mathbf{y}_k \in R^m$, $\mathbf{x}_k \in R^n$, and $\mathbf{v}_k$ is assumed to be a zero-mean white sequence with covariance R_k.

The measurement update problem is as follows: given the SR factors $V_{k+1/k}$ and $\Lambda^{1/2}_{k+1/k}$ of $P_{k+1/k}$, where $V_{k+1/k}$ is the eigenvectors matrix at time t_{k+1} given observations up to and including time t_k, $\Lambda^{1/2}_{k+1/k}$ is the diagonal matrix of the square roots of the eigenvalues, and $P_{k+1/k} = V_{k+1/k}\Lambda_{k+1/k}V^T_{k+1/k}$, compute the *a posteriori* square root factors, namely: $V_{k+1/k+1}$ and $\Lambda^{1/2}_{k+1/k+1}$, without computing explicitly the covariance $P_{k+1/k}$ (i.e. no "squaring" of the factors is permitted in order not to lose the square root characteristics of the method). The solution to the measurement update problem is summarized as the main result in the next theorem.

Theorem 1. *Measurement update of the spectral factors*

Given: the time propagated factors $V_{k+1/k}$ and $\Lambda^{1/2}_{k+1/k}$ of $P_{k+1/k}$ (which is assumed to be positive definite), the measurement matrix H_{k+1} and the non-singular measurement noise covariance matrix R_{k+1}, define the augmented matrix A_{k+1} as follows:

$$A_{k+1} \triangleq [V_{k+1/k}\Lambda^{-1/2}_{k+1/k} \,|\, H^T_{k+1}R^{-T/2}_{k+1}], \tag{2.2}$$

and perform a singular value decomposition of it:

$$A_{k+1} = Y_{k+1}[\Sigma_{k+1}\,|\,0]\,Z^T_{k+1}, \tag{2.3}$$

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then the measurement updated spectral factors $V_{k+1/k+1}$ and $\Lambda_{k+1/k+1}$ are obtained as the following result of the SVD:

$$V_{k+1/k+1} = Y_{k+1} \quad (2.4a)$$

$$\Lambda^{1/2}_{k+1/k+1} = \Sigma^{-1}_{k+1}. \quad (2.4b)$$

(Y_{k+1} is an $n \times n$ orthogonal matrix whose columns are the eigenvectors of $A_{k+1}A^{T}_{k+1}$, and Σ_{k+1} is an $n \times n$ diagonal matrix whose entries are the singular values of A_{k+1}, i.e. the positive square roots of the eigenvalues of $A_{k+1}A^{T}_{k+1}$.)

Proof. Write the measurement update equation in the "information form":

$$P^{-1}_{k+1/k+1} = P^{-1}_{k+1/k} + H^{T}_{k+1}R^{-1}_{k+1}H_{k+1}. \quad (2.5)$$

Using the spectral factors of P, (2.5) may be written as:

$$V_{k+1/k+1}\Lambda^{-1}_{k+1/k+1}V^{T}_{k+1/k+1} = V_{k+1/k}\Lambda^{-1}_{k+1/k}V^{T}_{k+1/k} + H^{T}_{k+1}R^{-1}_{k+1}H_{k+1} \quad (2.6)$$

and using the definition of A_{k+1} from (2.2) it is easy to verify that

$$V_{k+1/k+1}\Lambda^{-1}_{k+1/k+1}V^{T}_{k+1/k+1} = A_{k+1}A^{T}_{k+1}. \quad (2.7)$$

In (2.7) A_{k+1} is replaced by its SVD factors (2.3) to get

$$V_{k+1/k+1}\Lambda^{-1}_{k+1/k+1}V^{T}_{k+1/k+1} = Y_{k+1}\Sigma^{2}_{k+1}Y^{T}_{k+1}, \quad (2.8)$$

from which the result follows.

Note that the new update algorithm is free of explicit equations, a fact that may be advantageous in certain implementations. Rather, the algorithm is based upon the SVD technique. The latter is an extremely important tool of modern numerical linear algebra (Klema and Laub, 1976), today it also plays an important role in linear control theory. As mentioned before, several efficient and reliable subroutines which compute the SVD are in use today in mathematical packages as LINPACK (Dongarra *et al*., 1979), EISPACK (Garbow *et al*., 1977) and others. Probably one of the best known algorithms for computing the SVD is the Golub–Reinsch algorithm (Golub and Reinsch, 1970). This efficient algorithm was shown to be numerically stable. The stability stems from the fact that it is based on the orthogonal Householder and Givens transformations, which are famous for their numerical stability and accuracy (Golub and Van Loan, 1983). The new measurement update algorithm, which employs the SVD, inherits those excellent characteristics.

To complete the SR update algorithm the algorithm for updating the state estimate must be specified. The latter is the ordinary KF algorithm; consequently, what is left to be determined is how to compute the filter gain using the square root factors. This may be done in two ways, as follows.

Define the matrix $M_{k+1/k+1}$ as

$$M_{k+1/k+1} = W^{T}_{k+1/k+1}H^{T}_{k+1}, \quad (2.9)$$

where the square root $W_{k+1/k+1}$ is

$$W_{k+1/k+1} = V_{k+1/k+1}\Lambda^{1/2}_{k+1/k+1}. \quad (2.10)$$

Then it is easy to see that the Kalman gain, based on the *a posteriori* factors, is

$$K_{k+1} = W_{k+1/k+1}M_{k+1/k+1}R^{-1}_{k+1}. \quad (2.11)$$

Note that in the conventional KF algorithm there is no direct way of obtaining the gain using the *a posteriori* covariance matrix, $P_{k+1/k+1}$, because the computation of $P_{k+1/k+1}$ requires already the knowledge of the gain.

An alternative formula for the computation of the gain based on the *a priori* factors is obtained as follows. Define the matrix $M_{k+1/k}$ as

$$M_{k+1/k} = W^{T}_{k+1/k}H^{T}_{k+1}, \quad (2.12)$$

where the *a priori* square root $W_{k+1/k}$ is defined analogously to (2.10).

The Kalman gain K_{k+1} is then

$$K_{k+1} = W_{k+1/k}M_{k+1/k}(M^{T}_{k+1/k}M_{k+1/k} + R_{k+1})^{-1}. \quad (2.13)$$

The decision on whether to use (2.11) or (2.13) to compute K depends on the values of R_{k+1} and $M^{T}_{k+1/k}M_{k+1/k} + R_{k+1}$. For example, $(M^{T}_{k+1}M_{k+1} + R_{k+1})$ might be algorithmically singular in (2.13), in which case the use of (2.11) may be the only way of computing the gain reliably; or, on the other hand, R_{k+1} may be very "small" in (2.11), so that the gain computation using (2.11) might lead to amplification of numerical noise in the state measurement update, while the use of (2.13) poses no problem.

Finally, in the next theorem an algorithm for the measurement update when a suboptimal gain is used is outlined.

Theorem 2. Measurement update of the spectral factors using general (not necessarily optimal) gain

Given the time propagated factors $V_{k+1/k}$ and $\Lambda^{1/2}_{k+1/k}$ of $P_{k+1/k}$, the measurement matrix H_{k+1}, the non-singular measurement noise covariance R_{k+1} and the gain matrix $\bar{K}_{k+1}$ (which may be suboptimal), define the augmented matrix B_{k+1} as

$$B_{k+1} \triangleq [(I - \bar{K}_{k+1}H_{k+1})V_{k+1/k}\Lambda^{1/2}_{k+1/k} \mid \bar{K}_{k+1}R^{1/2}_{k+1}] \quad (2.14)$$

and perform a singular value decomposition of it

$$B_{k+1} = \tilde{Y}_{k+1}[\Xi_{k+1} \mid 0]\tilde{Z}^{T}_{k+1}, \quad (2.15)$$

then the updated factors are

$$V_{k+1/k+1} = \tilde{Y}_{k+1} \quad (2.16a)$$

$$\Lambda^{1/2}_{k+1/k+1} = \Xi_{k+1}. \quad (2.16b)$$

The derivation of this algorithm follows from the Joseph form update (Maybeck, 1981):

$$P_{k+1/k+1} = [I - \bar{K}_{k+1}H_{k+1}]P_{k+1/k}[I - \bar{K}_{k+1}H_{k+1}]^{T} + \bar{K}_{k+1}R_{k+1}\bar{K}^{T}_{k+1} \quad (2.17)$$

along the same lines of the proof of Theorem 1.

In Oshman and Bar-Itzhack (1985b) three well known examples are presented where it is shown analytically that the above discrete $V - \Lambda$ measurement update algorithm has the accuracy and numerical stability characteristics of an SR routine.

Having obtained the algorithms for the gain computation, the new measurement update algorithm is complete. To form a complete estimator, this result has to be tailored to a time update algorithm. This is done next.

3. *$V - \Lambda$ square root filtering*

In this section two versions of the $V - \Lambda$ SR filter are presented. They are the continuous/discrete and the discrete/discrete filters.

A. *The continuous/discrete SR filter*. The time update of the state estimate is similar to that of the ordinary KF, which poses no problem that the SR algorithms are intended to solve. The time update of the covariance matrix consists of the slution of the following Lyapunov equation

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^{T} + Q(t), \quad (3.1a)$$

with the initial condition

$$P(t_k) = P_{k/k}. \quad (3.1b)$$

Its $V - \Lambda$ solution is a special case of the $V - \Lambda$ solution of the matrix Riccati equation (Oshman and Bar–Itzhack, 1985a). The resulting algorithm is presented in Table 1, where the complete continuous/discrete algorithm is summarized.

Next a discrete/discrete estimation algorithm is derived, in which the time update stage is also performed in a discrete manner.

TABLE 1. THE CONTINUOUS/DISCRETE $V-\Lambda$ FILTER ALGORITHM

System model	$\mathbf{x}(t) = F(t)\mathbf{X}(t) + G(t)\mathbf{w}(t)$; $E\{\mathbf{w}(t)\} = \mathbf{0}$; $E\{\mathbf{w}(t)\mathbf{w}(s)^{\mathrm{T}}\} = Q(t)\delta(t-s)$
Measurement model	$\mathbf{v}(t_k) = H(t_k)\mathbf{x}(t_k) + \mathbf{v}(t_k)$; $E\{\mathbf{v}(t_k)\} = \mathbf{0}$; $E\{\mathbf{v}(t_k)\mathbf{v}(t_l)^{\mathrm{T}}\} = R(t_k)\delta_{kl}$
Initial conditions	$\hat{\mathbf{x}}_0 = E[\mathbf{x}(0)]$, $P_0 = E[(\mathbf{x}(0) - \hat{\mathbf{x}}_0)(\mathbf{x}(0) - \hat{\mathbf{x}}_0)^{\mathrm{T}}]$
State estimate time update	$\dot{\hat{\mathbf{x}}}(t) = F(t)\hat{\mathbf{x}}(t)$; $t \in [t_k, t_{k+1}]$ $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_{k/k}$; $\hat{\mathbf{x}}_{k+1/k} = \hat{\mathbf{x}}(t_{k+1})$.
$V-\Lambda^{1/2}$ time update	$\begin{cases} \dot{S}(t) = \frac{1}{2}\Gamma(t)S(t)^{-1} & S(t) \triangleq \Lambda(t)^{1/2} \\ \dot{V}(t) = V(t)\Omega(t) \end{cases}$ $\Gamma(t) = \mathrm{diag}[\gamma_{11}(t), \gamma_{22}(t), \ldots, \gamma_{nn}(t)]$ $\gamma_{ii}(t) = \mathbf{v}_i^{\mathrm{T}}(t)T_{ii}(t)\mathbf{v}_i(t)$ $T_{ii}(t) = s_i^2(t)[F(t) + F(t)^{\mathrm{T}}] + G(t)Q(t)G(t)^{\mathrm{T}}$ $\Omega_{qi}(t) = \begin{cases} \dfrac{\gamma_{iq}(t)}{s_i(t)^2 - s_q(t)^2} & \text{for } \left\lvert\dfrac{\gamma_{iq}(t)}{s_i(t)^2 - s_q(t)^2}\right\rvert < \Omega_{\max} \\ 0 & \text{for } s_i(t) = s_q(t) \\ \Omega_{\max}\,\mathrm{sign}\left\{\dfrac{\gamma_{iq}(t)}{s_i(t)^2 - s_q(t)^2}\right\} & \text{for } \left\lvert\dfrac{\gamma_{iq}(t)}{s_i(t)^2 - s_q(t)^2}\right\rvert \geqslant \Omega_{\max} \end{cases}$ $\gamma_{iq}(t) = \mathbf{v}_q^{\mathrm{T}}(t)T_{iq}(t)\mathbf{v}_i(t)$ $T_{iq}(t) = s_i(t)^2F(t) + s_q(t)^2F(t)^{\mathrm{T}} + G(t)Q(t)G(t)^{\mathrm{T}}$
$V-\Lambda^{1/2}$ measurement update	$A_{k+1} \triangleq [V_{k+1/k}\Lambda_{k+1/k}^{-1/2} \mid H_{k+1}R_{k+1}^{-\mathrm{T}/2}]$ $A_{k+1} \xrightarrow{\mathrm{SVD}} Y_{k+1}[\Sigma_{k+1} \mid 0]Z_{k+1}^{\mathrm{T}}$ Read: $V_{k+1/k+1} = Y_{k+1}$; $\Lambda_{k+1/k+1}^{1/2} = \Sigma_{k+1}^{-1}$
State estimate measurement update	$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + K_{k+1}[\mathbf{y}_{k+1} - H_{k+1}\hat{\mathbf{x}}_{k+1/k}]$
Kalman gain matrix	*a posteriori*: $K_{k+1} = W_{k+1/k+1}M_{k+1/k+1}R_{k+1}^{-1}$ where $W_{k+1/k+1} = V_{k+1/k+1}\Lambda_{k+1/k+1}^{1/2}$, $M_{k+1} = W_{k+1/k+1}^{\mathrm{T}}H_{k+1}^{\mathrm{T}}$ or *a priori*: $K_{k+1} = W_{k+1/k}M_{k+1/k}(M_{k+1/k}^{\mathrm{T}}M_{k+1/k} + R_{k+1})^{-1}$ where $W_{k+1/k} = V_{k+1/k}\Lambda_{k+1/k}^{1/2}$, $M_{k+1/k} = W_{k+1/k}^{\mathrm{T}}H_{k+1}^{\mathrm{T}}$

B. *Discrete/discrete SR filter.* As in the continuous/discrete version, the time propagation of the state estimate using the ordinary KF algorithm poses no problem which the SR algorithms are intended to solve. Therefore, the ordinary KF algorithm for propagating the state estimate can be used either in its discrete or even in its continuous version. Again, the problem is in propagating the $V-\Lambda$ factors of the covariance matrix. The algorithm for doing that is formulated in Theorem 3.

Theorem 3. *Time update of the spectral factors*

Given the measurement updated factors $V_{k/k}$ and $\Lambda_{k/k}^{1/2}$ of $P_{k/k}$, the state transition matrix Φ_k, the input gain matrix G_k and the process noise covariance Q_k, define the augmented matrix $\bar{A}_k$ as

$$\bar{A}_k = [\Phi_k V_{k/k}\Lambda_{k/k}^{1/2} \mid G_k Q_k^{1/2}] \tag{3.2}$$

and use an SVD routine to decompose $\bar{A}_k$ into

$$\bar{A}_k = \bar{Y}_k[\bar{\Sigma}_k \mid 0]\bar{Z}_k^{\mathrm{T}}, \tag{3.3}$$

then the time updated factors of $P_{k+1/k}$ are

$$V_{k+1/k} = \bar{Y}_k \tag{3.4a}$$

$$\Lambda_{k+1/k}^{1/2} = \bar{\Sigma}_k. \tag{3.4b}$$

Proof. The time update equation, using the spectral factors, is

$$V_{k+1/k}\Lambda_{k+1/k}V_{k+1/k}^{\mathrm{T}} = \Phi_k V_{k/k}\Lambda_{k/k}V_{k/k}^{\mathrm{T}}\Phi_k^{\mathrm{T}} + G_kQ_kG_k^{\mathrm{T}}. \tag{3.5}$$

Using the definition of $\bar{A}_k$ from (3.2), (3.5) becomes:

$$V_{k+1/k}\Lambda_{k+1/k}V_{k+1/k}^{\mathrm{T}} = \bar{A}_k\bar{A}_k^{\mathrm{T}} \tag{3.6}$$

which, replacing $\bar{A}_k$ by its SVD factors (3.3), gives

$$V_{k+1/k}\Lambda_{k+1/k}V_{k+1/k}^{\mathrm{T}} = \bar{Y}_k\bar{\Sigma}_k^2\bar{Y}_k^{\mathrm{T}} \tag{3.7}$$

and the result follows.

When the foregoing discrete time propagation is augmented with the discrete measurement update algorithm presented in Section 2, a complete discrete/discrete $V-\Lambda$ SR filter is obtained. This algorithm is summarized in Table 2.

C. *Discussion.* As mentioned before, relying upon the SVD technique, the new $V-\Lambda$ filters suffer from a higher computational cost than other existing SR filtering algorithms. Thus, using the Golub and Reinsch SVD algorithm (1970), the time update and the measurement update of the $V-\Lambda$ factors in

TABLE 2. THE DISCRETE/DISCRETE $V-\Lambda$ FILTER ALGORITHM

System model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + G_k \mathbf{w}_k,\quad E\{\mathbf{w}_k\} = \mathbf{0};\quad E\{\mathbf{w}_k \mathbf{w}_l^{\mathrm{T}}\} = Q_k \delta_{kl}$
Measurement model	$\mathbf{y}_k = H_k \mathbf{x}_k + \mathbf{v}_k, E\{\mathbf{v}_k\} = \mathbf{0};\quad E\{\mathbf{v}_k \mathbf{v}_l^{\mathrm{T}}\} = R_k \delta_{kl}$
Initial conditions	$\hat{\mathbf{x}}_0 = E[\mathbf{x}(o)],\quad P_0 = \mathbf{E}[(\mathbf{x}(0) - \hat{\mathbf{x}}_0)(\mathbf{x}(0) - \hat{\mathbf{x}}_0)^{\mathrm{T}}]$
State estimate time update	$\hat{\mathbf{x}}_{k+1/k} = \Phi_k \hat{\mathbf{x}}_{k/k}$
SVD processor	$A_k \overset{\mathrm{SVD}}{\rightarrow} Y_k[\Sigma_k \mid 0] Z_k^{\mathrm{T}}$ where $A_k \in R^{n,n+r},\quad Y_k \in R^{n,n},\quad \Sigma_k \in R^{n,n}$. input: A_k; output: Y_k, Σ_k
$V-\Lambda^{1/2}$ time update	Use the SVD Processor with the input: $A_k = [\Phi_k V_{k/k} \Lambda_{k/k}^{1/2} \mid G_k Q_k^{1/2}] \in R^{n,n+p}$ Read: $V_{k+1/k} = Y_k$ $\Lambda_{k+1/k}^{1/2} = \Sigma_k$
$V-\Lambda^{1/2}$ measurement update	Use the SVD Processor with the input: $A_k = [V_{k+1/k} \Lambda_{k+1/k}^{-1/2} \mid H_{k+1}^{\mathrm{T}} R_{k+1}^{-\mathrm{T}/2}] \in R^{n,n+m}$ Read the results: $V_{k+1/k+1} = Y_{k+1}$ $\Lambda_{k+1/k+1}^{1/2} = \Sigma_{k+1}^{-1}$
State estimate measurement update	$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + K_{k+1}[\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1/k}]$
Kalman gain matrix	*a posteriori* $K_{k+1} = W_{k+1/k+1} M_{k+1/k+1} R_{k+1}^{-1}$, where $W_{k+1/k+1} = V_{k+1/k+1} \Lambda_{k+1/k+1}^{1/2}$ $M_{k+1} = W_{k+1/k+1}^{\mathrm{T}} H_{k+1}^{\mathrm{T}}$ or *a priori* $K_{k+1} = W_{k+1/k} M_{k+1/k} (M_{k+1/k}^{\mathrm{T}} M_{k+1/k} + R_{k+1})^{-1}$, where $W_{k+1/k} = V_{k+1/k} \Lambda_{k+1/k}^{1/2}$ $M_{k+1/k} = W_{k+1/k}^{\mathrm{T}} H_{k+1}^{\mathrm{T}}$

Note: the algorithms appearing in Tables 1 and 2 assume white measurement noise that is uncorrelated with the process noise.

the discrete/discrete filter require both $O(6n^3)$ flops* (Golub and Van Loan, 1983), while Bierman's U–D algorithm requires $O(n^2)$ flops for the measurement update of the U–D factors (Bierman, 1976) and $O(1.5n^3)$ flops for the time update (Thornton and Bierman, 1975). However, in view of the new hardware development, mentioned in the introduction, it is believed that the additional computation required by the new algorithm will be of no concern.

It is of interest to note that the $V-\Lambda$ SR filters presented above are hybrid type filters, which utilize alternately the covariance mode (in the time update stage) and the information mode (in the measurement update stage). Thus, because of the operation in both modes, the new filters possess the advantages of the covariance and information filters. These advantages are: the ability to cope with the case of infinite initial covariance (no initial information), the efficiency of the covariance formulation in processing time updates and the efficiency of the information formulation in processing measurement updates. Moreover, because of the duality between the discrete time update of the covariance factors and the discrete measurement update of the information matrix factors, the fact that the $V-\Lambda$ filter operates in both modes implies algorithmic equivalence between the procedures used in the two stages of the filter (as can be seen by comparing the measurement update of Section 2 with the time update of this section). This equivalence introduces a saving factor in the implementation of the filter, because both stages actually use the same algorithm. This fact is also valuable for its simplification of the error analyses of specific implementations.

* A flop is roughly the amount of work needed to carry out the FORTRAN statement: $S = S + A(I, K)*B(K, J)$ (Golub and Van Loan, 1983).

4. *Filtering example*

In this section results of a simple filtering example are presented. The purposes of this are (a) to demonstrate that the new $V-\Lambda$ algorithm works satisfactorily and (b) to demonstrate the superior numerical stability and accuracy of the new algorithm when compared to the conventional KF algorithm.

Example 4.1. The dynamic system is that of a simplified single channel Inertial Navigation System (INS) error model which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta p\\ \delta V\\ \phi\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -g\\ 0 & R^{-1} & 0\end{bmatrix}\begin{bmatrix}\delta p\\ \delta v\\ \phi\end{bmatrix} + \begin{bmatrix}0\\ \varepsilon_a\\ \varepsilon_g\end{bmatrix}, \tag{4.1a}$$

where earth gravity, g, is 9.81 m s^{-2} and the reciprocal of earth radius, R^{-1}, is 0.157E–6 m^{-1}. The states δp, δv and ϕ are, respectively, position, velocity and tilt errors. Finally, ε_a and ε_g are, respectively, accelerometer and gyro generated white noise components whose mean is zero. The measurement equation is taken as

$$\mathbf{z}_k = \begin{bmatrix}0.4 & 1.0 & 0.0\\ 0.0 & 1.0 & 0.0\end{bmatrix}\begin{bmatrix}\delta p\\ \delta v\\ \phi\end{bmatrix}_k + \mathbf{n}_k, \tag{4.1b}$$

where $\mathbf{n}_k$ is the measurement zero mean white noise, whose covariance is

$$R_k = \mathrm{diag}[0.008 \quad 0.008].$$

Equation (4.1a) is propagated at 0.1-s steps, thus the discrete dynamic equation obtained from (4.1a) is

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \boldsymbol{\varepsilon}_k,$$

where

$$\Phi_k = \begin{bmatrix} 1.0 & 1.0 & -0.04905 \\ 0.0 & 1.0 & -0.981 \\ 0.0 & 0.157\text{E-7} & 1.0 \end{bmatrix} \quad \mathbf{x}_k = \begin{bmatrix} \delta p_k \\ \delta v_k \\ \phi_k \end{bmatrix} \quad \boldsymbol{\varepsilon}_k = \begin{bmatrix} 0 \\ \varepsilon_{ak} \\ \varepsilon_{gk} \end{bmatrix}.$$

The vector $\boldsymbol{\varepsilon}_k$ is a zero mean white sequence, whose covariance matrix Q_k is chosen as

$$Q_k = \text{diag}[0.0 \quad 0.2\text{E-9} \quad 0.15\text{E-15}].$$

The initial estimation error covariance matrix is chosen to be

$$P_0 = \text{diag}[0.25\text{E} + 5 \quad 0.12\text{E} + 5 \quad 0.12\text{E} + 5].$$

The initial state is

$$\mathbf{x}_0^{\mathrm{T}} = [5.0 \quad 1.0 \quad 0.273\text{E-1}]$$

and its initial estimate is chosen to be

$$\hat{\mathbf{x}}_0^{\mathrm{T}} = [1.0 \quad 0.5 \quad 0.005].$$

The discrete $V - \Lambda$ filter and the discrete conventional KF are both used to obtain an estimate $\hat{\mathbf{x}}_k$ of $\mathbf{x}_k$. The simulation is run taking measurements at a rate of $10\,\text{s}^{-1}$. The $V - \Lambda$ filter is used in single precision (SP), while the conventional filter is run both in single and in double precision (DP). As expected, the results obtained by the $V - \Lambda$ filter in SP are almost identical to those obtained by the DP version of the KF. On the other hand, the SP version of the conventional filter results in negative entries on the covariance main diagonal, which in turn results in a very large estimation error during the first updates. In Fig. 1 the time history of the standard deviation of the estimation error of δp is shown, as obtained by the three filters. As can be seen from this figure, the conventional filter loses all numerical significance in the first time points, which results in negative variances (the corresponding square roots are plotted as negative values in Fig. 1). In Fig. 2 the time history of the absolute value of the estimation error of δp is shown.

Finally, note that when the initial covariance is set to

$$P_0 = \text{diag}[0.25\text{E} + 9 \quad 0.12\text{E} + 9 \quad 0.12\text{E} + 9],$$

the conventional KF cannot handle this relatively large initial covariance because of an algorithmic singularity which occurs in the term $[H_0 P_0 H_0^{\mathrm{T}} + R_0]$ (which has to be inverted during the computation of the Kalman gain). Using the $V - \Lambda$ SR filter, however, such initialization problems are not encountered, since the covariance factors are updated independently of the gain, and the gain itself can, this time, be computed using the *a posteriori* factors (see the discussion in Section 3).

5. *Conclusions*

In this paper two new $V - \Lambda$ square root filter algorithms were presented. Only the case of vectorial updating was presented since the scalar updating is merely a special case of the former. It should be noted that in some other SR routines one can perform scalar updating only, while here one has the choice of performing either one.

A simple filtering example was also presented which indicated that the new filter algorithm works satisfactorily, and demonstrated its superiority over the conventional KF algorithm.

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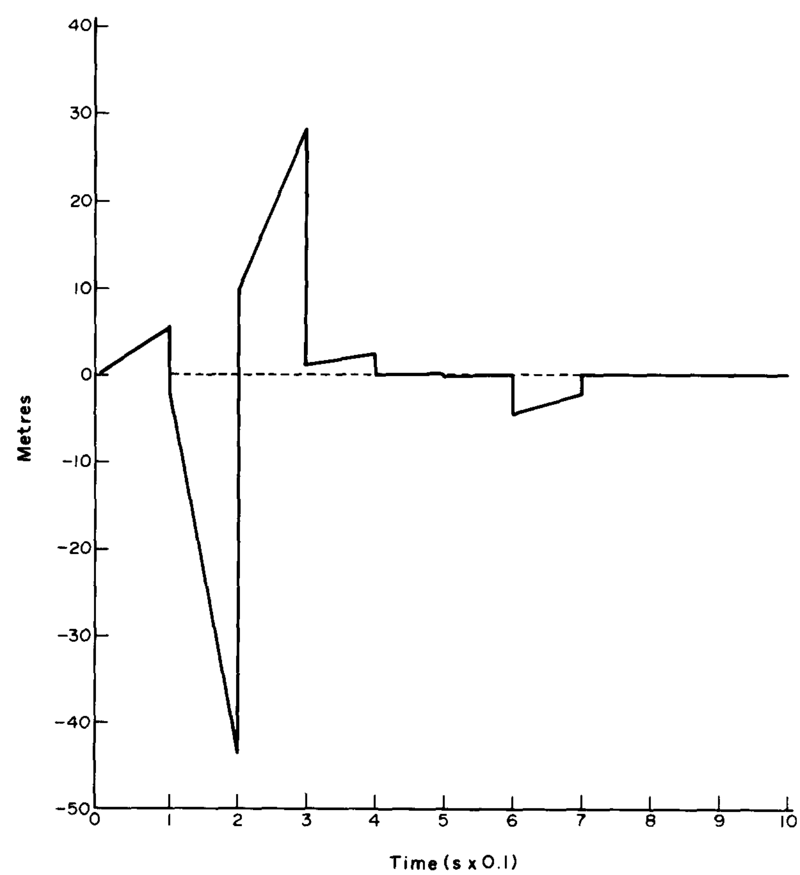

FIG. 1. Standard deviation of the estimation error of δp. (The solid line is for the conventional KF in single precision. The broken line is for the $V - \Lambda$ filter in single precision and for the conventional KF in double precision.)

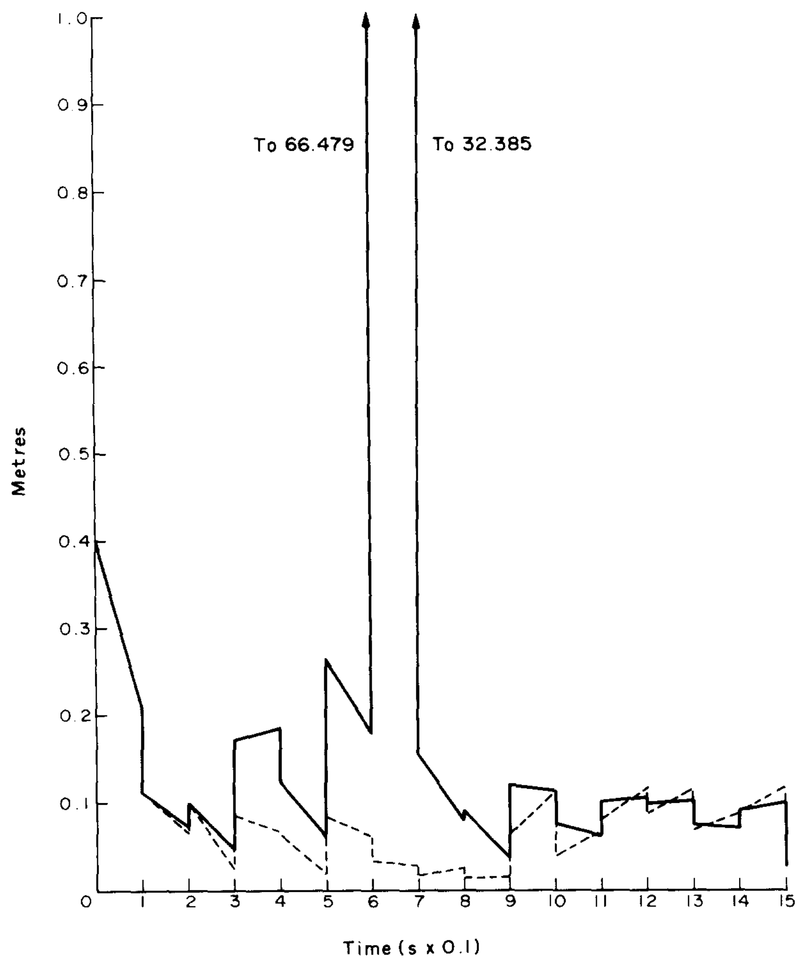

FIG. 2. Absolute value of the estimation error of δp. (The solid line is for the conventional KF in single precision. The broken line is for the $V-\Lambda$ filter in single precision and for the conventional KF in double precision.)

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