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Modal Control of Piezolaminated Anisotropic Rectangular Plates Part 1: Modal Transducer Theory

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Selective modal transducers are developed for piezolaminated anisotropic plate systems that are capable of sensing and exciting any specified set of vibrational modes according to a specified set of modal participation factors. Transduction of selected modes is accomplished through combining the effect of three piezolaminate pairs whose piezoelectric fields are varied spatially. Each coupled pair contains a single layer located anywhere strictly above the reference plane, which is complemented by a second layer colocated below the reference plane. Piezoelectric constitutive properties associated with each layer in a given couple must be identical, whereas the constitutive properties of all three couples must be uniquely different. If all selective modal transducer layers are formed from the same stock material, the stock material must be piezoelectrically biaxial and the skew angles of all couples must be unique. Individual actuator inputs must be proportional to a common control function or, conversely, the sensed output must be a weighted sum of the measurements acquired by individual layers. An algorithm is presented that dictates how the piezoelectric field strength of each selective modal transducer layer must be varied spatially and is an explicit function of piezoelectric constants, mode shapes, and designer-chosen modal participation factors. Selective modal transducers for orthotropic systems are shown to require three piezolaminate layers rather than three coupled pairs.

I. Introduction

WITHIN the past decade several vibration control techniques have been developed for simple beam and plate systems that utilize distributed piezoelectric transducers formed from polyvinylidine fluoride (PVDF).[1–3] PVDF actuators have been designed whose spatially varying piezoelectric field properties were exploited to provide for the simultaneous control of all modes or the selective control of desired modal subsets in cantilevered and simply supported beams.[4] Miller and Hubbard[5] developed a reciprocal sensor theory and subsequently incorporated PVDF sensors and actuators into multicomponent systems in which each component itself was a smart structural member.[6,7] Burke and Hubbard[8] developed a formulation for the control of thin elastic (Kirchhoff–Love) isotropic plates subject to most combinations of free, clamped, or pinned boundary conditions, in which the active elements were spatially varying biaxially polarized piezoelectric transducer layers. Lee[9] and Lee and Moon[10] generalized the classical laminate theory[11] to include the effect of laminated piezoelectric layers and, thus, to provide a theoretical framework for the distributed transduction of bending, torsion, shearing, shrinking, and stretching in flexible anisotropic plates. Miller et al.[12] subsequently employed Lyapunov's second method to derive a general active vibration suppression control design methodology for anisotropic laminated piezoelectric plates.

The aforementioned vibration control strategies for both beams and plates share several common limitations. Although all of these methods reduce the vibration control task to a selection of individual piezolaminae field functions, none offers a general method for determining those field functions so as to ensure active vibration suppression. A poor choice in piezofield functions, although guaranteed not to destabilize the structure through the active addition of vibrational energy, may extract little or no vibrational energy from the system. Furthermore, often the designer is concerned with suppressing vibrations in only a certain modal subset. The generalized function approach to choosing spatial field functions,[4,8] although adequate in certain scenarios for guaranteeing some measure of active energy extraction from all modes, generally will not be able to provide a means to selectively target a specific modal subset. Finally, most methodologies mentioned have been exclusive to isotropic systems and are thus incompatible for use with orthotropic and anisotropic aeroelastic structures commonly encountered. Ultimately these limitations would be best answered through the development of a selective modal control (SMC) methodology for anisotropic plates in which the designer optimally utilizes the available piezolaminae so as to most effectively realize any admissible performance objective.

As a first step toward realizing an SMC design methodology, a selective modal transducer (SMT) theory is herein developed. SMTs are a class of transducers that are capable of sensing and exciting any specified set of vibrational modes of an anisotropic plate in a selectively weighted fashion. The transduction of selected modal subsets of an anisotropic plate is accomplished through combining the effect of six piezolaminae whose piezoelectric field distributions vary spatially. Orthotropic plate SMTs are shown to be possible with as few as three layers. Design criteria are identified that lead to an algorithm for determining the specific piezoelectric field distributions and feedback gains required of each layer in the composite transducer. The SMC methodology is then presented in our companion paper.[13]

II. System Description

A. Geometry

Figure 1 describes the geometry of the general system under consideration. A rectangular anisotropic plate with exactly N piezoelectrically active laminate layers is considered. Contrary to what may be implicitly assumed from the figure, transducer layers may be located anywhere in the structure. The length dimensions of the system in the x and y directions are denoted as L_a and L_b, respectively. Each piezoelectric layer may be independently anisotropic, although typically the active layers are either transversely isotropic or else their mechanical stiffness relative to the substrate allows their anisotropy to be neglected. The material properties within each lamina are assumed continuous. The bounded domain containing the system under consideration is denoted as A and its boundary is denoted as Γ.

Each piezolamina is assumed to be uniformly coated on both surfaces with negligibly thin conductive electrode layers. An electrical field applied across the thickness of the kth piezolamina generates

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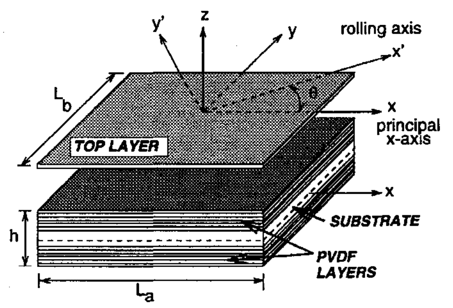

Fig. 1 Geometry of general piezoelectric laminate system.

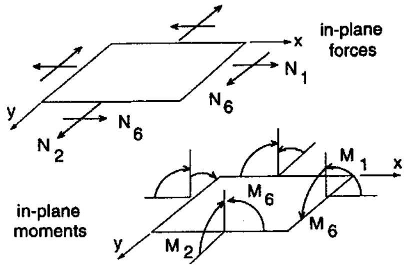

Fig. 2 Geometry of in-plane force and moment resultants.

a mechanical strain state whose principal (dominant rolling) axes are signified by x_k' and y_k'. Conversely, mechanical strain in the x_k'–y_k' plane induces an electrical field. Although it is assumed that the applied electrical field is quasistatically uniform with respect to the dominant plane of the lamina, the magnitude of electromechanical transduction may be spatially varied by means of doping the copolymer with PZT powder[14] or through repoling.[9] The moduli of each layer need not be constant throughout the dominant plane of the plate. A positive poling direction of each layer is defined as outwardly normal to the reference plane of the system. The reference plane itself may be arbitrarily located, although it is typically assigned to the structural midplane. In an orthotropic structure, however, the reference plane is designated as the neutral plane. The strain displacement relationships for each laminate are assumed to be governed by the Kirchhoff–Love approximation in which displacements of the laminae are related to each other linearly through the thickness direction. Finally, the dominant rolling axis x_k' of each piezoelectric laminate may be rotated from the principal x axis through a skew angle θ, defined in a positive right-hand sense about the z axis and illustrated in Fig. 1. The introduction of such rotations induces a torsional effect in piezoelectric film actuators and leads to the detection of shear strain in compatible sensors.[9]

B. Equations of Motion

Expressed in terms of resultant forces and moments, the equations of motion of the general system described in Fig. 1 are[11]

$$\begin{aligned} \rho h u_{tt} - (N_1)_x - (N_6)_y &= 0 \\ \rho h v_{tt} - (N_2)_y - (N_6)_x &= 0 \\ \rho h w_{tt} - (M_1)_{xx} - 2(M_6)_{xy} - (M_2)_{yy} &= 0 \end{aligned} \tag{1}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the respective axial displacements of the system reference plane in the x and y principal directions and $w(x, y, t)$ is the transverse displacement of the reference plane. Subscripts indicate partial differentiation. The equivalent density ρ is defined as

$$\rho \triangleq \sum_{k=1}^{N} \frac{\rho_k h_k}{h} \tag{2}$$

where ρ_k and h_k are the mass densities and thicknesses of each lamina in the composite structure and h is the total thickness. The mass density and thickness of each layer are assumed to be constant. The in-plane resultant forces ($\boldsymbol{N} \triangleq [N_1 \;\; N_6 \;\; N_2]'$) and moments ($\boldsymbol{M} \triangleq [M_1 \;\; M_6 \;\; M_2]'$) are defined geometrically in Fig. 2. In mathematical terms they represent the net difference between mechanically and piezoelectrically induced force and moment resultants[9]

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_M - \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_P \tag{3}$$

Introducing the linear homogeneous differential operators $\mathcal{D}$ and $\mathcal{E}$ defined as

$$\{\mathcal{D}, \mathcal{E}\} \triangleq \left\{ \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x^2} & 2\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix}', \begin{bmatrix} -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x^2} & 2\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix}' \right\} \tag{4}$$

the mechanically induced force and moment resultants[11] may be restated such that

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_M = -\boldsymbol{K}_a \mathcal{E} \boldsymbol{x} \tag{5}$$

where

$$\boldsymbol{K}_a \triangleq \begin{bmatrix} A_{11} & A_{16} & A_{12} & B_{11} & B_{16} & B_{12} \\ A_{16} & A_{66} & A_{26} & B_{16} & B_{66} & B_{26} \\ A_{12} & A_{26} & A_{22} & B_{12} & B_{26} & B_{22} \\ B_{11} & B_{16} & B_{12} & D_{11} & D_{16} & D_{12} \\ B_{16} & B_{66} & B_{26} & D_{16} & D_{66} & D_{26} \\ B_{12} & B_{26} & B_{22} & D_{12} & D_{26} & D_{22} \end{bmatrix} \tag{6}$$

and

$$\boldsymbol{x} \triangleq [u \quad v \quad w]' \tag{7}$$

A_{ij}, B_{ij}, and D_{ij} are the constitutive material constants that characterize the mechanical stress–strain behavior of the composite system. The piezoelectrically induced force and moment resultants[9] may be likewise restated,

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_P = \sum_{k=1}^{N} \boldsymbol{e}^k V^k \tag{8}$$

where $V^k = V^k(t)$ is the spatially independent driving voltage applied across the kth lamina, whose piezoelectric behavior is characterized by the vector of piezoelectric constitutive parameters

$$\boldsymbol{e}^k \triangleq \left[e_{31} \quad e_{36} \quad e_{32} \quad z^k e_{31} \quad z^k e_{36} \quad z^k e_{32} \right]' \tag{9}$$

for which z^k is the height of the kth piezolamina midplane above the composite reference plane. The magnitude of these parameters may be spatially varied by doping the copolymer with PZT powder[14] or subjecting the film to a repoling process.[9] These processes cause the piezoelectric strength at a given location on the film to vary while preserving the relative proportionality between the e_{31}, e_{32}, and e_{36} parameters. Assuming that the piezoelectric effect is constant through the thickness, the doping and repoling processes are then accounted for through the introduction of a dimensionless, spatially varying piezoelectric field distribution function $\Lambda^k(x, y)$. The field function is defined such that the piezoelectric field vector $\boldsymbol{e}^k$ is equivalently expressed in the form

$$\boldsymbol{e}^k = \boldsymbol{e}_0^k \Lambda^k \tag{10}$$

where e_0^k contains the values of e_{31}^k, e_{32}^k, and e_{36}^k at the point of maximum electromechanical transduction so that Λ^k is normalized, i.e., the maximum magnitude of Λ^k is unity. For convenience a vector e_*^k is also defined such that

$$e_*^k \triangleq \begin{bmatrix} (e_{31})_0^k & (e_{36})_0^k & (e_{32})_0^k \end{bmatrix}' \tag{11}$$

Substituting Eq. (10) into Eq. (8) yields

$$\begin{bmatrix} N \\ M \end{bmatrix}_P = \sum_{k=1}^{N} e_0^k \Lambda^k V^k \tag{12}$$

Substitution of Eqs. (5) and (12) into Eq. (3) then yields the identity

$$\begin{bmatrix} N \\ M \end{bmatrix} = -K_a \mathcal{E} x - \sum_{k=1}^{N} e_0^k \Lambda^k V^k \tag{13}$$

Returning to the equations of motion [Eq. (1)] and applying Eqs. (4), (7), and (6) leads to

$$x_{tt} + \frac{1}{\rho h}\mathcal{D}'(K_a \mathcal{E} x) = -\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} e_0^k \Lambda^k V^k\right) \tag{14}$$

Defining the mass-normalized stiffness operator $\mathcal{K}$ as

$$\mathcal{K} \triangleq (1/\rho h)\mathcal{D}'(K_a \mathcal{E}) \tag{15}$$

the equations of motion are then rendered in the attractive form

$$x_{tt} + \mathcal{K} x = -\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} e_0^k \Lambda^k V^k\right) \tag{16}$$

Structural damping may be introduced into Eq. (16) through consideration of a (mass normalized) damping operator $\mathcal{C}$, which is proportional to the operator $\mathcal{K}$ by a positive factor c_0 (Ref. 15), whereas damping losses due to the mechanical interaction of the plate with the medium in which it is contained (atmospheric drag, etc.) are introduced through a constant b_0,

$$\mathcal{C} = b_0 \mathcal{I} + c_0 \mathcal{K} \tag{17}$$

where $\mathcal{I}$ is the identity operator. The damping operator is then included in the equations of motion such that

$$x_{tt} + \mathcal{C} x_t + \mathcal{K} x = -\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} e_0^k \Lambda^k V^k\right) \tag{18}$$

Although representative of a fully anisotropic piezolaminated system, the equations of motion as stated revert to a general orthotropic system by setting all B_{ij} to zero. A specially orthotropic system emerges if, likewise, A_{16}, A_{26}, D_{16}, and D_{26} are set to zero. Similarly, the equations of motion for a general isotropic plate emerge through appropriate choices in the remaining constitutive constants. Beam equations result when v is set to zero and the displacements w and u are independent of y.

The preceding system of equations is subject to the boundary conditions stated in Table 1[9,11] where the transverse shear force resultants Q_1 and Q_2 are defined as

$$(Q_1) = [(M_1)]_x + [(M_6)]_y \tag{19}$$

$$(Q_2) = [(M_6)]_x + [(M_2)]_y \tag{20}$$

Table 1 Boundary conditions

$x = -L_a/2, L_a/2$	$y = -L_b/2, L_b/2$
(N_1) or u	(N_2) or v
(N_6) or v	(N_6) or u
(Q_1) or w	(Q_2) or w
(M_1) or w_x	(M_2) or w_y
(M_6) or w_y	(M_6) or w_x

The force and moment resultants are defined in Eq. (13). The boundary conditions are in a form compatible with Poisson's derivation.[16] Realize, however, that the system described by Eq. (16) can at most accommodate four boundary conditions per edge.[16] Kirchhoff showed that the twisting moment and shear force boundary conditions are related and, accordingly, two of the boundary conditions on each edge may be combined into a single condition.[16] The Poisson form will nonetheless prove to be extremely useful in the ensuing analysis.

The domain of definition for the operator $\mathcal{K}$ (and thus $\mathcal{C}$) is now explicitly defined. Let $H(A)$ denote the Hilbert space of all real-valued piecewise continuous functions whose inner product and norm are, respectively, defined as

$$\langle g, h \rangle = \iint_A g' h \, \mathrm{d}A \tag{21}$$

and

$$\|g\| = \langle g, g \rangle^{\frac{1}{2}} \tag{22}$$

for any $g(x, y), h(x, y) \in H(A)$. Denoting the order of $\mathcal{K}$ as n, let any admissible set of boundary conditions given in Table 1 be described in terms of linear spatial differential operators $\mathcal{B}_i$ of maximum order $n - 1$ such that

$$\mathcal{B}_i x = 0 \qquad \text{on} \qquad \Gamma, \qquad i = 1, \ldots, n \tag{23}$$

Let S be the set of all functions g for which $\mathcal{B}_i g = 0$ on Γ and such that g and all of its n derivatives are in $H(A)$. In the ensuing development the admissible set of boundary conditions to be considered are such that $\mathcal{K}$ is rendered regular on S and has an inverse defined by a Green's function. Note that any set of boundary conditions that does not permit rigid body motions automatically satisfies this criterion. A procedure for explicitly determining the Green's function inverse is given in Ref. 17.

C. Sensor Equation

The current accumulated on the surface electrode of the kth lamina due to the mechanical displacement of the laminates is[9]

$$i^k(t) = \iint_A \Big[e_{31}^k u_{xt} + e_{32}^k v_{yt} + e_{36}^k (u_{yt} + v_{xt}) - e_{31}^k z^k w_{xxt} - e_{32}^k z^k w_{yyt} - 2 e_{36}^k z^k w_{xyt} \Big] \, \mathrm{d}A \tag{24}$$

where $i^k(t)$ is the current measured through the kth electrode. Applying the definitions given for the differential operator $\mathcal{E}$ [Eq. (4)], x [Eq. (7)], and e^k [Eq. (10)], Eq. (24) collapses to

$$i^k(t) = -\iint_A (\mathcal{E} x_t)' e_0^k \Lambda^k \, \mathrm{d}A \tag{25}$$

III. Selective Modal Actuator Theory

In this section a general theory and design methodology is presented that allows the designer to selectively excite each and every mode of a general anisotropic piezolaminated plate typified in Fig. 1 according to a prespecified set of modal participation factors. Consider the following set of SMT construct conditions.

Condition C1: Exactly n transducer layers are located strictly above the reference plane and exactly n transducers are located strictly below the reference plane ($N = 2n$).

Condition C2: There are at least six piezoelectrically active layers.

Condition C3: For each layer above the reference plane there exists a layer below the reference plane such that $\{z^k = -z^{k+n}\}_{k=1}^n$.

Condition C4: Layers located at heights z^k and z^{k+n} both are associated with an identical piezoproperty vector e_*^k.

Condition C5: The piezoproperty vectors $\{e_*^k\}_{k=1}^n$ associated with at least three layers above (and hence also below) the reference plane are different. When the same sample of PVDF is used throughout, e_{31}^0 ($\theta^k = 0$ deg) $\neq e_{32}^0(\theta^k = 0$ deg) and the skew angles of at least three laminae above (and hence below) the plane must be different in the range $-90 \leq \theta^k < 90$ deg.

The following lemma is now introduced.

Lemma 1: Let $\boldsymbol{R} \in \mathcal{R}^{6,6}$ be the matrix defined as

$$\boldsymbol{R} \triangleq \sum_{k=1}^{N} \boldsymbol{e}_0^k \left(\boldsymbol{e}_0^k\right)' \tag{26}$$

Then, if C1–C5 hold, $\boldsymbol{R}$ is invertible. Furthermore, $\boldsymbol{R}$ can be written as

$$\boldsymbol{R} = 2\sum_{k=1}^{n} \begin{bmatrix} \boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' & \boldsymbol{0} \\ \boldsymbol{0} & (z^k)^2 \boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' \end{bmatrix} \tag{27}$$

Proof: Recalling from C1 that $N = 2n$ and substituting Eq. (11) into Eq. (10), $\boldsymbol{e}_0^k = [(\boldsymbol{e}_*^k)' \;\; z^k(\boldsymbol{e}_*^k)']'$ and, thus,

$$\boldsymbol{R} = \sum_{k=1}^{2n} \begin{bmatrix} \boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' & z^k\boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' \\ z^k\boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' & (z^k)^2 \boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' \end{bmatrix} \tag{28}$$

Satisfying C1, C3, and C4 then transforms $\boldsymbol{R}$ into the form of Eq. (27), which may equivalently be written as

$$\boldsymbol{R} = 2\begin{bmatrix} \sum_{k=1}^{n} \boldsymbol{e}_*^k\left(\boldsymbol{e}_*^k\right)' & \boldsymbol{0} \\ \boldsymbol{0} & \sum_{k=1}^{n} \left(z^k\boldsymbol{e}_*^k\right)\left(z^k\boldsymbol{e}_*^k\right)' \end{bmatrix} \tag{29}$$

C2 and C5 are seen to be necessary for the invertibility of the submatrices located on the diagonal of $\boldsymbol{R}$ based on the following postulate[18]:

Postulate 1: Given a set of column vectors $\{\boldsymbol{r}_k : \boldsymbol{r}_k \in \mathcal{R}^m\}$, the matrix $\boldsymbol{R}_0 \triangleq \sum_{k=1}^{n} \boldsymbol{r}_k\boldsymbol{r}_k'$ is invertible if and only if there exist at least m linearly independent vectors in the set $\{\boldsymbol{r}_k\}_{k=1}^{n}$.

The consequence of Postulate 1 is that the submatrices on the diagonal of the rightmost expression in Eq. (29) are invertible only if $n \geq 3$ and at least three elements of each vector subset $\{\boldsymbol{e}_*^k\}_{k=1}^{n}$ and $\{z^k\boldsymbol{e}_*^k\}_{k=1}^{n}$ are unique. C1 ensures that all z^k are nonzero, and the physical geometry ensures that all z^k are different. C2 and the first part of C5 then cause the elements of each vector subset to be independent.

The latter part of C5 pertains to the event in which the same sample of piezoelectric material is to be used throughout in constructing each active structural layer. In Appendix A it is shown that such a case requires that the sample material be piezoelectrically biaxial $[e_{31}^0(\theta^k = 0\text{ deg}) \neq e_{32}^0(\theta^k = 0\text{ deg})]$ and the skew angles of laminae above (and likewise below) the plane must be different in the range $-90 \leq \theta^k < 90$ deg. Obeying this constraint causes the piezoelectric field properties of each of the n piezolaminae above (and likewise below) the reference plane to be uniquely different with respect to the principal geometric directions. □

Another lemma is now stated.

Lemma 2: Consider an anisotropic rectangular plate containing N piezolaminae whose equations of motion are given by Eq. (18). Let the time bound control input $V^k(t)$ of each piezolamina be proportional to an identical time-dependent control function $V_a(t)$ such that $V^k(t) = g_0^k V_a(t)$. Assume that C1–C5 are satisfied. Then, if the piezoelectric field distribution functions of each active layer are given by

$$\Lambda^k = \left(1/g_0^k\right)\left(\boldsymbol{e}_0^k\right)'\boldsymbol{R}^{-1}\boldsymbol{K}_a\mathcal{E}\bar{\phi}(x,y) \tag{30}$$

where $\bar{\phi}$ is a weighted sum of eigenfunctions ϕ_j and modal participation factors α_j such that

$$\bar{\phi} \triangleq \sum_{j=1}^{\infty} \alpha_j\phi_j \tag{31}$$

while $\boldsymbol{R}$ is defined in Eq. (27), and the scaling factor g_0^k is defined as

$$g_0^k = \max_{(x,y)\in A} \left|\left(\boldsymbol{e}_0^k\right)'\boldsymbol{R}^{-1}\boldsymbol{K}_a\mathcal{E}\bar{\phi}(x,y)\right| \tag{32}$$

the equations of motion of the plate reduce to the form

$$\boldsymbol{x}_{tt} + \mathcal{C}\boldsymbol{x}_t + \mathcal{K}\boldsymbol{x} = -V_a(t)\mathcal{K}\bar{\phi} \tag{33}$$

Proof: Recalling the equations of motion [Eq. (18)] and the definition $\mathcal{K} \triangleq (1/\rho h)\mathcal{D}'(\boldsymbol{K}_a\mathcal{E})$, Lemma 2 is proven if the conditions as stated cause the equality

$$\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} \boldsymbol{e}_0^k\Lambda^k V^k\right) = V_a(t)\frac{1}{\rho h}\mathcal{D}(\boldsymbol{K}_a\mathcal{E}\bar{\phi}) \tag{34}$$

to be true. Letting $V^k(t) = g_0^k V_a(t)$, the left side of Eq. (34) is then transformed such that

$$\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} \boldsymbol{e}_0^k\Lambda^k V^k\right) = V_a(t)\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} g_0^k\boldsymbol{e}_0^k\Lambda^k\right) \tag{35}$$

Substituting Eq. (30) into the right-hand side of Eq. (35) then yields

$$\frac{1}{\rho h}\mathcal{D}'\left(\sum_{k=1}^{N} \boldsymbol{e}_0^k\Lambda^k V^k\right) = V_a(t)\frac{1}{\rho h}\mathcal{D}'\left[\sum_{k=1}^{N} \boldsymbol{e}_0^k\left(\boldsymbol{e}_0^k\right)'\right]\boldsymbol{R}^{-1}\boldsymbol{K}_a\mathcal{E}\bar{\phi} \tag{36}$$

Since C1–C5 are satisfied, Lemma 1 holds so that $\boldsymbol{R} = \sum_{k=1}^{N} \boldsymbol{e}_0^k(\boldsymbol{e}_0^k)'$ is invertible and may be equivalently written in the form of Eq. (27). Substituting Eq. (26) into Eq. (36) then yields Eq. (34). □

The main result is now given.

Theorem 1: Consider an anisotropic rectangular plate containing N piezolaminae whose equations of motion are given by Eq. (18). Let the time bound control input $V^k(t)$ of each piezolamina be proportional to an identical time-dependent control function $V_a(t)$ such that $V^k(t) = g_0^k V_a(t)$. Assume that C1–C5 are satisfied. Then, if the piezoelectric field distribution functions of each active layer are given by Eq. (30) where $\bar{\phi}$ is defined in Eq. (31), $\boldsymbol{R}$ is defined in Eq. (27), and g_0 is defined in Eq. (32), the equations of motion of the plate reduce to the form

$$\ddot{q}_m + (b_0 + c_0\lambda)\dot{q}_m + \lambda_m q_m = -\alpha_m\lambda_m V_a(t) \tag{37}$$

for all integers $m > 0$ where α_m, λ_m, and $q_m(t)$ are the modal participation factor, eigenvalue, and generalized modal coordinate associated with the mth mode, respectively.

Proof: Since the suppositions stated in the theorem satisfy Lemma 2, Eq. (18) may be equivalently expressed in the form of Eq. (33). The distributed forces acting on the plate are therefore expressed in the form $\boldsymbol{f} = -V_a(t)\mathcal{K}\boldsymbol{x}_q$. Consider the following proposition, proven in Appendix B.

Proposition 1: If the distributed forces acting on the plate can be expressed as $\boldsymbol{f} = -V_a(t)\mathcal{K}\boldsymbol{x}_q$ for some $\boldsymbol{x}_q \in S$, then the operators $\mathcal{C}$ and $\mathcal{K}$ are self-adjoint.

Remark: The proposition implies that the self-adjointness of both operators is contingent upon the choice of $\boldsymbol{f}$. Realize that self-adjointness is defined with respect to any two functions $\boldsymbol{x}_p, \boldsymbol{x}_q \in S$ [see Eq. (B1), Appendix B] that, by definition, satisfy all boundary conditions. Since the boundary conditions as stated in Table 1 are directly related to the applied distributed forces as is evidenced by Eq. (13), the self-adjointness of $\mathcal{C}$ and $\mathcal{K}$ is contingent on the form that $\boldsymbol{f}$ assumes.

Inman[19] restated the result of Caughey and O'Kelly[20] in the following form.

Theorem 2: Let $\boldsymbol{x}_{tt} + \mathcal{C}\boldsymbol{x}_t + \mathcal{K}\boldsymbol{x} = \boldsymbol{f}(x,y,t)$ be the equations of motion of a general system excited by a distributed force $\boldsymbol{f}$. Then, if $\mathcal{C}$ and $\mathcal{K}$ commute and are self-adjoint on S, and if each operator has an inverse defined by a Green's function, then the solution to the governing equation may be written as the uniformly convergent series

$$\boldsymbol{x}(x,y,t) = \sum_{j=1}^{\infty} \phi_j(x,y)q_j(t) \tag{38}$$

where the set $\{\phi_j(x,y)\}_{j=1}^{\infty}$ are the orthonormal eigenfunctions of $\mathcal{K}$ that are identical to the eigenfunctions of $\mathcal{C}$.

It has already been presupposed that the boundary conditions render $\mathcal{K}$ (and thus $\mathcal{C}$) such that it has a Green's function inverse. The fact that the operators commute, i.e., $\mathcal{CK} = \mathcal{KC}$, follows trivially from Eq. (17). In view of Proposition 1, the system under discussion satisfies all conditions contained in the Theorem 2. Orthonormality is defined in the sense that[17]

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} \tag{39}$$

where δ_{ij} is the Kronecker delta function. Since $\mathcal{K}\phi_j = \lambda_j \phi_j$, where λ_j is the eigenvalue corresponding to the jth eigenfunction,

$$\langle \phi_i, \mathcal{K}\phi_j \rangle = \lambda_j \delta_{ij} \tag{40}$$

Substituting Eq. (38) into Eq. (33) then yields

$$\sum_{j=1}^{\infty}[\phi_j \ddot{q}_j + (b_0 + c_0\mathcal{K})\phi_j \dot{q}_j + \mathcal{K}\phi_j q_j] = -V_a(t)\sum_{j=1}^{\infty}\alpha_j \mathcal{K}\phi_j \tag{41}$$

Taking the inner product of the mth eigenfunction ϕ_m with each side of Eq. (41) and applying Eqs. (39) and (40) then yields Eq. (37), thus completing the proof of Theorem 1. □

Remark: Equation (37) indicates that any given structural mode may be excited as long as the modal participation factor belonging to that mode is nonzero. Since the system response is an infinite sum of modal responses, the ability to selectively weight the contribution of any given mode to the total system response implies complete controllability. Hence, the following corollary is stated.

Corollary 1: When all suppositions of Theorem 1 are satisfied, the system is completely controllable.

Theorem 1 represents an anisotropic piezolaminated plate selective modal actuator (SMA) design methodology. The SMT construct conditions C1–C5 are obeyed. Mode shapes belonging to the target modal subset are determined and then assigned modal participation factors. Then using Eq. (30) as an algorithm for determining the piezoelectric field function for each layer and enforcing the condition $V^k(t) = g_0^k V_a(t)$, a modal actuator design is realized that is capable of exciting each mode according to its relative weighting.

In general, it will be advantageous to use as few layers as is necessary, which is six. Furthermore, in many applications it may be deemed preferable to use the same sample of PVDF to construct all active layers in the composite. In Appendix A it is shown that in such a case, all laminae above (and similarly below) the reference plane must be skewed at different angles with respect to the principal geometric directions. If the layers are chosen so as to preserve symmetry with respect to the principal x axis of the plate, then any combination of the angles {−60, 0, 60 deg} maximizes the determinant of the matrix $\boldsymbol{E}_0 \triangleq \sum_{k=1}^{3} \boldsymbol{e}_*^k(\boldsymbol{e}_*^k)'$. It is also shown in Appendix A that $\boldsymbol{E}_0$ is poorly conditioned when the ratio of e_{31}^0 to e_{32}^0 is less than 1.3.

IV. Selective Modal Sensor Theory

In this section a general theory and design methodology is presented that allows for the selective observation of each and every mode of a general anisotropic piezolaminated plate typified in Fig. 1 according to a prespecified set of modal participation factors. The design methodology is summarized in the following theorem.

Theorem 3: Consider an anisotropic rectangular plate containing N piezolaminae whose equations of motion are given by Eq. (18). Let the measured state $i_s(t)$ be formed from the weighted sum of the sensed currents of each individual lamina be such that $i_s(t) = \sum_{k=1}^{N} g_0^k i^k(t)$. Assume that C1–C5 are satisfied. Then, if the piezoelectric field distribution functions of each active layer are given by Eq. (30) such that $\bar{\phi}$ is defined in Eq. (31), $\boldsymbol{R}$ is defined in Eq. (27), and g_0^k is defined as in Eq. (32), the measured state reduces to the form

$$i_s(t) = -\rho h \sum_{j=1}^{\infty} \alpha_j \lambda_j \dot{q}_j(t) \tag{42}$$

where α_j, λ_j, and $\dot{q}_j$ are the modal participation factor, eigenvalue, and generalized modal velocity associated with the jth eigenfunction, respectively.

Proof: Letting $i_s(t) = \sum_{k=1}^{N} g_0^k i^k(t)$ as stipulated in the theorem transforms Eq. (25) into the form

$$i_s(t) = -\iint_A (\mathcal{E}x_t)'\left(\sum_{k=1}^{N} g_0^k \boldsymbol{e}_0^k \Lambda^k\right) \mathrm{d}A \tag{43}$$

Substituting Eq. (30) into Eq. (43) and applying Lemma 1 yields

$$i_s(t) = -\iint_A (\mathcal{E}x_t)' \boldsymbol{K}_a(\mathcal{E}\bar{\phi})\,\mathrm{d}A \tag{44}$$

It is assumed that the system, if excited, obeys all conditions specified in Theorem 2 so that the system response is specified by Eq. (38) and Eqs. (39) and (40) apply. Substituting Eq. (38) into Eq. (44) yields

$$i_s(t) = -\sum_{i=1}^{\infty} \dot{q}_i \iint_A (\mathcal{E}\phi_i)' \boldsymbol{K}_a(\mathcal{E}\bar{\phi})\,\mathrm{d}A \tag{45}$$

Recalling that $\bar{\phi} \triangleq \sum_{j=1}^{\infty} \alpha_j \phi_j$, Eq. (45) becomes

$$i_s(t) = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_j \dot{q}_i \iint_A (\mathcal{E}\phi_i)' \boldsymbol{K}_a(\mathcal{E}\phi_j)\,\mathrm{d}A \tag{46}$$

In Appendix B [Eq. (B17)] it is shown that for two (comparison) functions $\boldsymbol{x}_p, \boldsymbol{x}_q \in S$,

$$\langle \boldsymbol{x}_p, \mathcal{K}\boldsymbol{x}_q \rangle = \frac{1}{\rho h}\iint_A (\mathcal{E}\boldsymbol{x}_q)' \boldsymbol{K}_a(\mathcal{E}\boldsymbol{x}_p)\,\mathrm{d}A \tag{47}$$

Equations (40) and (47) then allow Eq. (46) to be written as Eq. (42), completing the proof. □

Remark: Equation (42) implies that the modal velocity corresponding to any given mode or subset of structural modes may be selectively measured, which leads naturally to the following corollary.

Corollary 2: When all suppositions of Theorem 3 are satisfied, the system is completely observable.

Having established Theorem 3, an anisotropic piezolaminated selective modal sensor theory has been defined. By obeying those conditions stipulated in the theorem, a sensor may be developed whose measurement is a selectively weighted sum of modal velocities. Moreover, the conditions stipulated in Theorem 3 are identical to those required by Theorem 1 with the exception that the modal actuator requires that $V^k(t) = g_0^k V_a(t)$ whereas the modal sensor requires that $i_s(t) = \sum_{k=1}^{N} g_0^k i^k(t)$. Note that piezoelectric transducers have already been employed successfully in practice as self-sensing actuation devices.[21,22] Theorems 1 and 3 may thus be implemented simultaneously on the same set of piezolaminae, yielding true SMTs.

V. Orthotropic Plate SMTs

A. System Description

If the composite system is orthotropic ($\boldsymbol{B} = 0$) so that there is no inherent mechanical coupling between bending and stretching motions, then the SMT design constraints thus far specified may be significantly relaxed. The geometry of the system now under consideration is assumed to be equivalent to that shown in Fig. 1, but the substrate is orthotropic and the reference plane is now assumed to be the composite neutral plane. The transverse displacement w, therefore, is not related to the axial displacements u and v. Hence, the equations of motion for the orthotropic system may be written in the form of two uncoupled equations, the first of which describes pure membrane stretching and shearing motions and the second of which describes pure bending and twisting motions

$$(\boldsymbol{x}_s)_{tt} + \mathcal{C}_s(\boldsymbol{x}_s)_t + \mathcal{K}_s \boldsymbol{x}_s = -\frac{1}{\rho h}\mathcal{D}_b'\left(\sum_{k=1}^{N} \boldsymbol{e}_*^k \Lambda^k V^k\right) \tag{48}$$

$$w_{tt} + \mathcal{C}_b w_t + \mathcal{K}_b w = -\frac{1}{\rho h}\mathcal{D}_s'\left(\sum_{k=1}^{N} z^k \boldsymbol{e}_*^k \Lambda^k V^k\right) \tag{49}$$

where

$$x_s \triangleq [u \quad v]' \tag{50}$$

$$\{\mathcal{K}_s, \mathcal{K}_b\} \triangleq \left\{ -\frac{1}{\rho h}\mathcal{D}_s'(A\mathcal{D}_s), \frac{1}{\rho h}\mathcal{D}_b'(D\mathcal{D}_b) \right\} \tag{51}$$

$$\{\mathcal{D}_s, \mathcal{D}_b\} \triangleq \left\{ \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}', \begin{bmatrix} \frac{\partial^2}{\partial x^2} & 2\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix}' \right\} \tag{52}$$

The material constant matrices $A, D \in \Re^{3,3}$ are defined in Eq. (6). The damping operators $\mathcal{C}_s$ and $\mathcal{C}_b$ are defined in the same manner as $\mathcal{C}$ was defined for the anisotropic plate problem [Eq. (17)].

In the context of general orthotropic plates the term stretching is assumed to include both axial stretching and in-plane shearing motions in its usage, whereas bending is assumed to include twisting motions as well. For future reference bending and stretching modes, which are explicit functions of $w(x, y)$ and $x_s(x, y)$, respectively, are referred to as $(\phi_j)_b$ and $(\phi_j)_s$. The following definitions are then introduced:

$$\{\bar{\phi}_s, \bar{\phi}_b\} \triangleq \left\{ \sum_{j=1}^{\infty} \alpha_j(\phi_j)_s, \sum_{j=1}^{\infty} \alpha_j(\phi_j)_b \right\} \tag{53}$$

B. Orthotropic SMT Theory

Consider the imposition of the following two SMT construct conditions.

Condition C6: There are at least three piezoelectrically active layers ($N \geq 3$).

Condition C7: The piezoproperty vectors $\{e_*^k\}_{k=1}^N$ associated with at least three layers are different. When the same sample of piezostock material is used throughout, $e_{31}^0(\theta^k = 0 \text{ deg}) \neq e_{32}^0(\theta^k = 0 \text{ deg})$ and the skew angles of at least three laminae must be different in the range $-90 \leq \theta^k < 90$ deg.

The following proposition follows directly from Postulate 1.

Proposition 2: If Conditions C6 and C7 are satisfied, then the matrix $[\sum_{k=1}^N e_*^k(e_*^k)']$ is invertible.

The orthotropic plate SMA design methodology is then a direct consequence of the following theorem.

Theorem 4: Consider an orthotropic plate containing n piezolaminae whose equations of motion are given by Eq. (48) for pure stretching motions and Eq. (49) for pure bending motions. Let the time bound control input $V^k(t)$ of each piezolamina be proportional to an identical time-dependent control function $V_a(t)$ such that $V^k(t) = g_0^k V_a(t)$. Assume that Conditions C6 and C7 are satisfied. If the piezoelectric field distribution functions of each active layer are given by

$$\Lambda^k = -\frac{1}{g_0^k}\left(e_*^k\right)' \left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]^{-1} A\mathcal{D}_s\bar{\phi}_s \tag{54}$$

in pure stretching such that g_0^k is defined as

$$g_0^k = \max_{(x,y)\in A} \left| \left(e_*^k\right)' \left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]^{-1} A\mathcal{D}_s\bar{\phi}_s \right| \tag{55}$$

or by

$$\Lambda^k = \frac{1}{g_0^k z^k}\left(e_*^k\right)' \left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]^{-1} D\mathcal{D}_b\bar{\phi}_b \tag{56}$$

in pure bending such that g_0^k is defined as

$$g_0^k = \max_{(x,y)\in A} \left| \frac{1}{z^k}\left(e_*^k\right)' \left(\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right)^{-1} D\mathcal{D}_b\bar{\phi}_b \right| \tag{57}$$

then the equations of motion of either system reduce to the form

$$\ddot{q}_m + (b_0 + c_0\lambda_m)\dot{q}_m + \lambda_m q_m = -\alpha_m\lambda_m V_a(t) \tag{58}$$

for all integers $m > 0$.

Proof: Returning respectively to Eqs. (48) and (49), assuming that $V^k(t) = g_0^k V_a(t)$, and applying Eqs. (54) and (56), respectively, leads to the following expressions:

$$(x_s)_{tt} + \mathcal{C}_s(x_s)_t + \mathcal{K}_s x_s = -V_a(t)\frac{1}{\rho h}\mathcal{D}_b'\left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]\left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]^{-1} A\mathcal{D}_s\bar{\phi}_s \tag{59}$$

$$w_{tt} + \mathcal{C}_b w_t + \mathcal{K}_b w = -V_a(t)\frac{1}{\rho h}\mathcal{D}_s'\left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]\left[\sum_{k=1}^N e_*^k\left(e_*^k\right)' \right]^{-1} D\mathcal{D}_b\bar{\phi}_b \tag{60}$$

Applying Proposition 2 and recalling Eq. (51), the preceding equations reduce to the form

$$(x_s)_{tt} + \mathcal{C}_s(x_s)_t + \mathcal{K}_s x_s = -V_a(t)\mathcal{K}_s\bar{\phi}_s \tag{61}$$

$$w_{tt} + \mathcal{C}_b w_t + \mathcal{K}_b w = -V_a(t)\mathcal{K}_b\bar{\phi}_b \tag{62}$$

Theorem 2 then allows the expressions to be transformed into Eq. (58). □

To complete the discussion, a sensor theorem is now presented whose proof follows that of Theorem 3.

Theorem 5: Consider an orthotropic plate containing n piezolaminae whose equations of motion are given by Eq. (48) for pure stretching motions and Eq. (49) for pure bending motions. Let the measured state $i_s(t)$ be formed from the weighted sum of the sensed currents of each individual lamina such that $i_s(t) = \sum_{k=1}^N g_0^k i^k(t)$. Assume that Conditions C6 and C7 are satisfied. Then, if the piezoelectric field distribution functions of each active layer are given by Eqs. (54) and (55) in pure stretching or by Eqs. (56) and (57) in pure bending, the measured state reduces to the form

$$i_s(t) = -\rho h \sum_{j=1}^{\infty} \alpha_j \lambda_j \dot{q}_j(t)$$

VI. Conclusions

SMTs are developed for piezolaminated anisotropic plates in which the contribution of each mode to the sensing or excitation of the composite anisotropic plate system may be selectively weighted. SMTs are formed by combining the piezoelectric effect of several piezolaminae. Anisotropic plate SMTs are shown to require three coupled piezolaminate pairs. Each coupled pair contains a single layer located anywhere strictly above the reference plane that is complemented by a second layer collocated below the reference plane. Piezoelectric field properties associated with each layer in a given couple must be identical, although the field properties of all three couples must be uniquely different. Individual actuator inputs must be proportional to a common control function or, conversely, the sensed output must be a weighted sum of the measurements acquired by individual layers. A piezoelectric field function algorithm was developed to provide a systematic methodology for choosing the required spatial geometries of each sublayer. SMTs for orthotropic systems are shown to require three piezolaminate layers rather than three coupled pairs. The design constraints may be relaxed further when the system under consideration is isotropic.

In our companion paper, an SMC theory is derived in which SMTs are optimally utilized to most effectively realize any admissible performance objective. A numerical example is given that serves to validate both the SMT and SMC theories.

Appendix A: Piezoelectric Field Matrix Invertibility

A. Skew Angle Influence

Although in principle it is possible to choose from several different samples of PVDF whose field vectors e_*^k are different (so as to satisfy Condition C5), in most situations it will more likely be preferable to use the same random sampling of commercially available PVDF film for all active layers in the structure. If the same standard sample of PVDF film is to be used throughout, one must consider that commercially available PVDF film is normally poled such that e^0_{36} is zero.[23] Thus, the formulation to follow will concern the common case in which the constitutive properties of all layers are assumed identical, and $e^0_{36} = 0$. The skewing of each lamina will then be considered so as to cause the piezofield properties of each layer to differ with respect to the principal geometric frame, and criteria will be established for the skew angle choices.

The piezoelectric constants of the kth layer are related to the skew angle θ^k of that layer as defined in Fig. 1 through the following relationship[24]:

$$\begin{bmatrix} e_{31} \\ e_{32} \\ e_{36} \end{bmatrix}^k_0 = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} e^0_{31} \\ e^0_{32} \\ e^0_{36} \end{bmatrix} \tag{A1}$$

where $m = \cos\theta^k$, $n = \sin\theta^k$, and $[e^0_{31} \;\; e^0_{32} \;\; e^0_{36}]$ without the subscript k are the values of the piezoelectric constants of any layer when $\theta^k = 0$. From Eq. (A1) it is evident that the necessary and sufficient condition to satisfy Postulate 1, namely, that each of the vectors in the set $\{e^k_*\}^3_{k=1}$ are linearly independent, is realized if and only if all of the skew angles in the set $\{\theta^k\colon -90 \le \theta^k < 90 \text{ deg}\}^3_{k=1}$ are different. If the film sample is piezoelectrically uniaxial, i.e., $e^0_{31} = e^0_{32}$, then $e^0_{36} = 0$ for any skew angle and Postulate 1 can not be satisfied.

B. Optimal Skew Angle Choices

Although any choice of skew angles in which all θ^k are different yields an invertible form of the matrix $\boldsymbol{E}_0$, where

$$\boldsymbol{E}_0 \triangleq \sum_{k=1}^{3} e^k_*\left(e^k_*\right)' \tag{A2}$$

it will be useful to know which choice in skew angles maximizes the determinant of $\boldsymbol{E}_0$ such that $\boldsymbol{E}_0$ is the least singular and, also, which range of skew angle choices will cause $\boldsymbol{E}_0$ to be well conditioned (with respect to inversion). For simplicity, the set of possible choices for θ^k in the analysis to follow will be constrained by the assumption that the layers are chosen so as to preserve symmetry with respect to the principal x axis of the plate.

Defining the quantities

$$\{e_s, e_p\} \triangleq \left\{\tfrac{1}{2}\left(e^0_{31} + e^0_{32}\right), \tfrac{1}{2}\left(e^0_{31} - e^0_{32}\right)\right\} \tag{A3}$$

then for an ordinary sample of PVDF for which $e^0_{36} = 0$,

$$(e_{31})^k_0 = e_s + e_p c_k \tag{A4}$$

$$(e_{32})^k_0 = e_s - e_p c_k \tag{A5}$$

$$(e_{36})^k_0 = e_p s^k \tag{A6}$$

where $c_k = \cos 2\theta^k$ and $s_k = \sin 2\theta^k$. The matrix $\boldsymbol{E}_0$ may then be expressed in the form

$$\boldsymbol{E}_0 = \sum_{k=1}^{3} \begin{bmatrix} (e_s + e_p c_k)^2 & e_s e_p s_k + e^2_p s_k c_k & e^2_s - e^2_p c^2_k \\ e_s e_p s_k + e^2_p s_k c_k & e^2_p s^2_k & e_s e_p s_k - e^2_p s_k c_k \\ e^2_s - e^2_p c^2_k & e_s e_p s_k - e^2_p s_k c_k & (e_s - e_p c_k)^2 \end{bmatrix} \tag{A7}$$

Letting $\{\theta_1, \theta_2, \theta_3\} = \{0, \gamma, -\gamma\}$ and solving for the determinant of $\boldsymbol{E}_0$ then yields

$$\|\boldsymbol{E}_0\| = 256 e^4_p e^2_s \cos^2\gamma \sin^6\gamma \tag{A8}$$

which is maximized at $\gamma = 60$ deg so as to imply a skew angle grouping that is any combination of the angles $\{-60, 0, 60 \text{ deg}\}$.

The ratio $N_r \triangleq \lambda_{\min}/\lambda_{\max}$ (the reciprocal of the condition number) is plotted in Fig. A1 for all integer values of the ratio $R \triangleq e^0_{31}/e^0_{32}$ between 1 and 10. N_r is independent of actual values chosen for e^0_{31} and e^0_{32} and is indicative of the sensitivity of Eq. (30) to roundoff error or to errors in the estimation of $\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\bar{\phi}$. A well-conditioned matrix is typically defined[25] as one in which $N_r \ge 10^{-2}$. Numerical results show that $N_r > 10^{-2}$ when $R > 1.3$. Clearly, the greater the ratio of e^0_{31} to e^0_{32}, the more favorably conditioned is the matrix $\boldsymbol{E}_0$. It is evident from the plot that N_r is maximized for $\gamma = 52$ deg. Figure A1 should be referenced when assessing one's choice in skew angles.

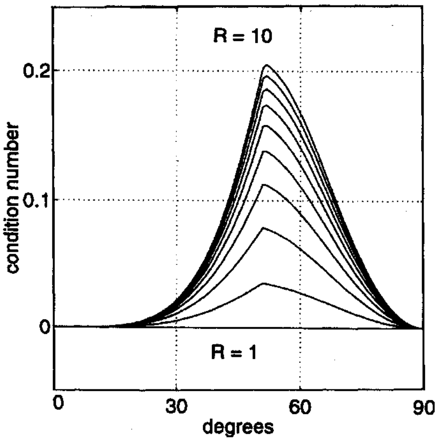

Fig. A1 Reciprocal condition number N_r as a function of the angle γ; plots are given for all integer values of R between 1 and 10.

Appendix B: Operator Self-Adjointness

Proposition 1 is now proven. Recall the proposition.

Proposition 1: If the distributed forces acting on the plate can be expressed as $\boldsymbol{f} = -V_a(t)\boldsymbol{\mathcal{K}}\boldsymbol{x}_q$ for some $\boldsymbol{x}_q \in S$, then the operators $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{K}}$ are self-adjoint.

Proof: Since $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{K}}$ are linear operators, it is evident from Eq. (17) that the operator $\boldsymbol{\mathcal{C}}$ is self-adjoint if and only if the operator $\boldsymbol{\mathcal{K}}$ is self-adjoint. $\boldsymbol{\mathcal{K}}$ is self-adjoint if

$$\langle \boldsymbol{x}_p, \boldsymbol{\mathcal{K}}\boldsymbol{x}_q\rangle = \langle \boldsymbol{x}_q, \boldsymbol{\mathcal{K}}\boldsymbol{x}_p\rangle \tag{B1}$$

for any arbitrary $\boldsymbol{x}_p, \boldsymbol{x}_q \in S$. In integral form the left side of Eq. (B1) becomes

$$\langle \boldsymbol{x}_p, \boldsymbol{\mathcal{K}}\boldsymbol{x}_q\rangle = \iint_A \boldsymbol{x}'_p\boldsymbol{\mathcal{K}}\boldsymbol{x}_q \,\mathrm{d}A \tag{B2}$$

Recalling the definition for $\boldsymbol{\mathcal{K}}$ [Eq. (15)], Eq. (B2) becomes

$$\langle \boldsymbol{x}_p, \boldsymbol{\mathcal{K}}\boldsymbol{x}_q\rangle = \frac{1}{\rho h}\iint_A \boldsymbol{x}'_p\boldsymbol{\mathcal{D}}'t(\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\boldsymbol{x}_q)\,\mathrm{d}A \tag{B3}$$

Given that $\boldsymbol{f} = -V_a(t)\,\boldsymbol{\mathcal{K}}\boldsymbol{x}_q$ and recalling that $\boldsymbol{\mathcal{K}} \triangleq (1/\rho h)\,\boldsymbol{\mathcal{D}}'(\boldsymbol{K}_a\boldsymbol{\mathcal{E}})$, it is evident from Eq. (34) that

$$\sum_{k=1}^{N} e^k_0 \Lambda^k V^k = V_a(t)\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\boldsymbol{x}_q \tag{B4}$$

The force and moment resultants as described by Eq. (13) are then written as

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = -\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\boldsymbol{x} - V_a(t)\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\boldsymbol{x}_q \tag{B5}$$

Suppose that the composite plate system under study is subject to a time-varying displacement field of the form $\boldsymbol{x} = \boldsymbol{x}_q q(t)$ where $q(t)$ represents an arbitrarily chosen time-dependent function. Applying Eq. (B5), the moment and force resultants due to the displacement field are then described as

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_q \triangleq -[q(t) + V_a(t)]\boldsymbol{K}_a\boldsymbol{\mathcal{E}}\boldsymbol{x}_q \tag{B6}$$

Substituting Eq. (B6) into Eq. (B3) then yields

$$\langle \boldsymbol{x}_p, \boldsymbol{\mathcal{K}}\boldsymbol{x}_q\rangle = -K(t)\iint_A \boldsymbol{x}'_p\boldsymbol{\mathcal{D}}'\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_q \mathrm{d}A \tag{B7}$$

where

$$K(t) \triangleq \frac{1}{\rho h[q(t) + V_a(t)]} \tag{B8}$$

Recalling Eq. (4),

$$\mathcal{D}'\begin{bmatrix} N \\ M \end{bmatrix}_q = \begin{bmatrix} [(N_1)_x + (N_6)_y]_q \\ [(N_6)_x + (N_2)_y]_q \\ [(M_1)_{xx} + 2(M_6)_{xy} + (M_2)_{yy}]_q \end{bmatrix} \tag{B9}$$

Expressing x_p as

$$x_p = \begin{bmatrix} u_p(x, y) \\ v_p(x, y) \\ w_p(x, y) \end{bmatrix} \tag{B10}$$

Eqs. (B9) and (B10) may be substituted into Eq. (B7) such that

$$\langle x_p, \mathcal{K}x_q \rangle = -K(t) \iint_A \{u_p[(N_1)_x + (N_6)_y]_q + v_p[(N_6)_x + (N_2)_y]_q + w_p[(M_1)_{xx} + 2(M_6)_{xy} + (M_2)_{yy}]_q\} \,\mathrm{d}A \tag{B11}$$

Integrating the preceding equation by parts yields

$$\langle x_p, \mathcal{K}x_q \rangle = K(t) \iint_A \{(N_1)_q(u_x)_p + (N_6)_q[u_y + v_x]_p + (N_2)_q(v_y)_p - (M_1)_q(w_{xx})_p - (M_6)_q(2w_{xy})_p - (M_2)_q(w_{yy})_p\} \,\mathrm{d}A + I_1 + I_2 \tag{B12}$$

where I_1 and I_2 are defined as the line integrals

$$I_1 = -K(t)\int_{-(L_b/2)}^{L_b/2} \{u_p(N_1)_q + v_p(N_6)_q + w_p[(M_1)_x + (M_6)_y]_q - (w_x)_p(M_1)_q - (w_y)_p(M_6)_q\}_{x=-L_a/2}^{x=L_a/2} \,\mathrm{d}y \tag{B13}$$

$$I_2 = -K(t)\int_{-(L_a/2)}^{L_a/2} \{u_p(N_6)_q + v_p(N_2)_q + w_p[(M_6)_x + (M_2)_y]_q - (w_x)_p(M_6)_q - (w_y)_p(M_2)_q\}_{x=-L_b/2}^{x=L_b/2} \,\mathrm{d}x \tag{B14}$$

Since $x_p, x_q \in S$ and, thus, satisfy all boundary conditions, the boundary integrals I_1 and I_2 are identically zero. Reconstructing Eq. (B12) in matrix form,

$$\langle x_p, \mathcal{K}x_q \rangle = -K(t)\iint_A [N_1 \quad N_6 \quad N_2 \quad M_1 \quad M_6 \quad M_2]_q(\mathcal{E}x_p) \,\mathrm{d}A \tag{B15}$$

or, identically,

$$\langle x_p, \mathcal{K}x_q \rangle = -K(t) \iint_A \begin{bmatrix} N \\ M \end{bmatrix}_q' (\mathcal{E}x_p) \,\mathrm{d}A \tag{B16}$$

Recalling Eqs. (B6) and (B8), the preceding expression may be recast in the form

$$\langle x_p, \mathcal{K}x_q \rangle = \frac{1}{\rho h} \iint_A (\mathcal{E}x_q)' K_a (\mathcal{E}x_p) \,\mathrm{d}A \tag{B17}$$

The integral in the preceding expression is a scalar quantity so that its transpose must equal itself. Realizing that K_a is a symmetric matrix, Eq. (B17) implies that

$$\langle x_p, \mathcal{K}x_q \rangle = \frac{1}{\rho h} \iint_A (\mathcal{E}x_p)' K_a (\mathcal{E}x_q) \,\mathrm{d}A = \langle x_q, \mathcal{K}x_p \rangle$$

and Proposition 1 is thus proven. □

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