JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 27, No. 3, May–June 2004

Optimal-REQUEST Algorithm for Attitude Determination

D. Choukroun,* I. Y. Bar-Itzhack,† and Y. Oshman‡
Technion—Israel Institute of Technology, Haifa 32000, Israel

REQUEST is a recursive algorithm for least-squares estimation of the attitude quaternion of a rigid body using vector measurements. It uses a constant, empirically chosen gain and is, therefore, suboptimal when filtering propagation noises. The algorithm presented here is an optimized REQUEST procedure, which optimally filters measurement as well as propagation noises. The special case of zero-mean white noises is considered. The solution approach is based on state-space modeling of the *K*-matrix system and uses Kalman-filtering techniques to estimate the optimal *K* matrix. Then, the attitude quaternion is extracted from the estimated *K* matrix. A simulation study is used to demonstrate the performance of the algorithm.

Introduction

ATTITUDE determination (AD) is a major component of spacecraft operation. The fundamental AD problem is to specify the orientation of the rigid-body spacecraft axes, expressed by a Cartesian coordinate frame $\mathcal{B}$ with respect to a given reference Cartesian coordinate system $\mathcal{R}$. When a physical vector, say the Earth magnetic field $\boldsymbol{x}$ is observed, a useful measurement model equation is

$$\boldsymbol{b} = \boldsymbol{b}^o + \boldsymbol{\delta b} = A\boldsymbol{r} + \boldsymbol{\delta b} \tag{1}$$

where A is the attitude matrix [also called the DCM matrix (see Ref. 1, p. 411)]; $\boldsymbol{b}^o$ and $\boldsymbol{r}$ are the projections of $\boldsymbol{x}$ on $\mathcal{B}$ and $\mathcal{R}$, respectively; and $\boldsymbol{\delta b}$ is the measurement error. Given two observations, it is possible to estimate A by means of a deterministic algorithm,[2] whereas a single observation is not sufficient to yield an unambiguous attitude matrix (see Ref. 3, p. 23).

In 1965, Wahba formulated the AD problem from vector observations as a least-squares estimation problem[4]:

Given the two sets of n vectors $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$, where $n \geq 2$, find the proper orthogonal matrix A, which brings the first set into the best least-squares coincidence with the second. That is, find A, which minimizes

$$\sum_{i=1}^{n} \|\boldsymbol{b}_i - A\boldsymbol{r}_i\|^2 \tag{2}$$

subject to the constraints $A^T A = \boldsymbol{I}_3$ and $\det(A) = 1$.

This problem, known as the Wahba problem, is a single-frame attitude determination problem; that is, it assumes that all vector measurements that are processed to estimate the attitude have been obtained at the same attitude. One family of solutions is concerned with the determination of that optimal matrix A itself (see Refs. 5–9 for earlier solutions and Refs. 10–13 for more recent methods), whereas another family is concerned with the determination of the corresponding optimal quaternion. In this paper we treat the latter.

The quaternion of rotation $\boldsymbol{q}$ is another popular attitude representation (see Ref. 1, pp. 758 and 759 and Refs. 14 and 15 for extensive surveys of attitude parameterizations). It is a unit vector in $\mathbb{R}^4$, related to A by

$$A(\boldsymbol{q}) = (q^2 - \boldsymbol{e}^T\boldsymbol{e})\boldsymbol{I}_3 + 2\boldsymbol{e}\boldsymbol{e}^T - 2q[\boldsymbol{e}\times] \tag{3}$$

(see Ref. 1, p. 414) where $\boldsymbol{e}$ and q are the vector and the scalar part, respectively, of $\boldsymbol{q}$; $\boldsymbol{I}_3$ is the 3×3 identity matrix; and the cross-product matrix $[\boldsymbol{e}\times]$ is defined by

$$[\boldsymbol{e}\times] \triangleq \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \tag{4}$$

In 1968, Davenport devised a method, known in the literature as the $\boldsymbol{q}$ method, for computing the optimal single-frame quaternion $\boldsymbol{q}$, that is, the quaternion which corresponds to the optimal A of Wahba's problem. As reported in Ref. 16, pp. A.1–A.11, the method is based on the following identity:

$$\frac{1}{2}\sum_{i=1}^{n} a_i \|\boldsymbol{b}_i - A\boldsymbol{r}_i\|^2 = 1 - \boldsymbol{q}^T K \boldsymbol{q} \tag{5}$$

where the K matrix in the right-hand side of Eq. (5) is obtained as follows. Consider a batch of n simultaneous observations $\boldsymbol{b}_i, \boldsymbol{r}_i$, $i = 1, 2, \ldots, n$, and the corresponding weights a_i, whose sum, with no loss of generality, equals one, that is,

$$\sum_{i=1}^{n} a_i = 1$$

Define the 3×3 matrices B and S, the 3×1 column matrix z, and the scalar σ as

$$B \triangleq \sum_{i=1}^{n} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T, \qquad S \triangleq B + B^T$$

$$z \triangleq \sum_{i=1}^{n} a_i \boldsymbol{b}_i \times \boldsymbol{r}_i, \qquad \sigma \triangleq tr(B) \tag{6}$$

where $tr(\cdot)$ denotes the trace operator. Then, the 4×4 symmetric matrix K of Eq. (5) is

$$K = \begin{bmatrix} S - \sigma\boldsymbol{I}_3 & z \\ z^T & \sigma \end{bmatrix} \tag{7}$$

Note that the trace of K equals zero. It is clear from Eq. (5) that the constrained minimization of Wahba's cost function is equivalent to the constrained maximization of a quadratic form of $\boldsymbol{q}$.

Presented as Paper 2001-4153 at the AIAA 41st Guidance, Navigation, and Control Conference, Montreal, 6–9 August 2001; received 23 December 2002; revision received 1 November 2003; accepted for publication 12 November 2003.

*Graduate Student, Faculty of Aerospace Engineering, currently Assistant Researcher Associate, University of California at Los Angeles, Department of Mechanical and Aerospace Engineering; dach@ucla.edu. Member AIAA.

†Sophie and William Shamban Professor, Member, Technion's Asher Space Research Institute, Faculty of Aerospace Engineering. Fellow AIAA.

‡Associate Professor, Member, Technion's Asher Space Research Institute, Faculty of Aerospace Engineering. Associate Fellow AIAA.

It is known that the problem of determining the stationary values of a quadratic form on the unit sphere leads to the solution of an eigenvalue problem.[17] As a result, the optimal quaternion estimate is the eigenvector of K that belongs to its largest positive eigenvalue. The highlights of the $\boldsymbol{q}$ method are that it optimally filters the measurement noises and that it requires neither linearization of some equation model nor an initial quaternion. This method has become a very popular AD technique and has inspired numerous algorithms. Keat (Ref. 16, pp. A.15–A.34) presented a modified version of the power method[18] for computing the sought eigenvector. Shuster and Oh[19] developed the QUEST (quaternion estimator) for this problem. QUEST is a very popular algorithm for a single-frame (SF) estimation of attitude quaternion. Mortari proposed two new SF algorithms, ESOQ[20] (estimator of the optimal quaternion) and ESOQ2.[21] However, these algorithms, as any SF attitude estimator, are memory-less algorithms, that is, the information contained in measurements of past attitudes is lost.

Methods have been devised in order to relax the requirement for the measurements to be acquired simultaneously. These methods require knowing the angular velocity of $\mathcal{B}$ with respect to $\mathcal{R}$. Markley[22] developed a batch algorithm using the body kinematics equation in terms of the attitude matrix $A(t)$. The highlights of this algorithm are that it provides an optimal predictor of $A(t)$ and that, in addition, it can estimate a set of constant disturbance parameters, such as gyro biases. Recursive estimators, which are more convenient than batch algorithms, have been developed, that perform sequential estimation of time-varying attitude,[23,24] as well as other parameters.[25]

In 1996, a discrete recursive algorithm named REQUEST that propagates and updates the K matrix was introduced.[24] This algorithm is summarized next. It is known (Ref. 1, p. 564) that the body angular motion can be described in terms of the attitude quaternion by the following difference equation:

$$\boldsymbol{q}_{k+1} = \Phi_k \boldsymbol{q}_k \tag{8}$$

where the transition matrix is computed as follows. Assume that $\boldsymbol{\omega}_k$ is constant during the time increment Δt, then the matrix Φ_k is expressed by (Ref. 1, pp. 511, 512)

$$\Phi_k = \exp(\Omega_k \Delta t) \tag{9}$$

where Ω_k is the following skew-symmetric matrix:

$$\Omega_k \triangleq \frac{1}{2}\begin{bmatrix} -[\boldsymbol{\omega}_k \times] & \boldsymbol{\omega}_k \\ -\boldsymbol{\omega}_k^T & 0 \end{bmatrix} \tag{10}$$

The matrix Φ_k is a 4×4 orthogonal matrix function of the angular velocity vector of the rotation of $\mathcal{B}$ with respect to $\mathcal{R}$, $\boldsymbol{\omega}_k$. Denote by $K_{i/j}$ the K matrix at time t_i, which is constructed from the measurements up to time t_j. According to the REQUEST algorithm, the propagation of the K matrix from t_k to t_{k+1} is given by

$$K_{k+1/k} = \Phi_k K_{k/k} \Phi_k^T \tag{11}$$

Given a single observation at time t_{k+1}, that is, $\boldsymbol{r}_{k+1}$ and $\boldsymbol{b}_{k+1}$, one can construct the corresponding incremental K matrix, denoted by δK_{k+1}, as follows. First define

$$B_{k+1} \triangleq a_{k+1}\boldsymbol{b}_{k+1} r_{k+1}^T, \qquad S_{k+1} \triangleq B_{k+1} + B_{k+1}^T$$

$$z_{k+1} \triangleq a_{k+1}\boldsymbol{b}_{k+1} \times r_{k+1}, \qquad \sigma_{k+1} \triangleq tr(B_{k+1}) \tag{12}$$

then compute δK_{k+1} as

$$\delta K_{k+1} = \frac{1}{a_{k+1}}\begin{bmatrix} S_{k+1} - \sigma_{k+1}\boldsymbol{I}_3 & z_{k+1} \\ z_{k+1}^T & \sigma_{k+1} \end{bmatrix} \tag{13}$$

where a_{k+1} is the scalar weighting coefficient of the $(k+1)$ observation. The update stage of REQUEST is of the form

$$K_{k+1/k+1} = (\rho_k m_k / m_{k+1}) K_{k+1/k} + (a_{k+1}/m_{k+1})\delta K_{k+1} \tag{14}$$

where m_k and m_{k+1} are scalar coefficients that keep the largest eigenvalue of $K_{k+1/k+1}$ close to one[24]; that is, $m_{k+1} = \rho_k m_k + a_{k+1}$, for $k = 0, 1, \ldots,$ and $m_0 = a_0$. The coefficient ρ_k is a fading memory factor (Ref. 26, pp. 285–288), which is equal to 1 if Φ_k is error free, and is otherwise set between 0 and 1 according to the propagation noise. (Although REQUEST can handle a batch of new measurements, for our purpose we consider only one new measurement.) Treating the propagation noise using the fading memory concept makes REQUEST suboptimal. The aim of the present paper is to introduce an algorithm, called Optimal-REQUEST, which is based on REQUEST and is able to optimally filter the propagation noise.

The remainder of this paper is organized as follows: The problem that is solved in the present work is stated in the next section. Then, we lay the foundation necessary for the development of an optimized REQUEST algorithm. The Optimal-REQUEST algorithm is presented in the following section. The filter performance is then illustrated through a numerical example. Finally, we present our conclusions.

Problem Statement

The choice of ρ_k is heuristic, making the filter REQUEST suboptimal. Moreover ρ_k is determined by considering the accuracy of the propagation stage only, disregarding the accuracy of the measurement.

We wish to optimize REQUEST by computing an optimal value of the parameter ρ_k in the update stage of REQUEST. The optimal value of ρ_k is that which yields an optimal blending of the a priori estimate of the K matrix $K_{k+1/k}$ and its new observation δK_{k+1}. Optimality is achieved by minimizing a proxy measure of the uncertainty in the a posteriori estimate of the K matrix. [An exact expression of the cost function to be minimized is defined later in Eqs. (57) and (58).]

Mathematical Model

Solution Approach

The approach to computing an optimal gain ρ_k^* consists of embedding REQUEST in the framework of Kalman filtering. With this purpose in mind, we modify the formulation of the attitude estimation problem. The central idea is that we would know the value of the true attitude quaternion at any time had we known the value of the true K matrix at any time. The true K matrix is the K matrix that does not contain any error, neither from vector measurement noise, nor from propagation noise. Because we cannot know the true K matrix, we propose to estimate it in some optimal way.

A notation that is consistent with that of REQUEST is used here. Denote by $K_{i/j}^o$ the true K matrix related to the attitude at t_i, which is based on the ideal noise-free vector measurements up to t_j. The estimate of this matrix is denoted by $K_{i/j}$. Similarly, the true K matrix that is based on the noise-free vector measurements acquired at t_i is denoted by δK_i^o. The measured K matrix, computed using the noisy vector measurements acquired at t_i, is denoted by δK_i.

We consider the true K matrix as a state-matrix variable and derive the dynamics and measurement equations that describe the state K-matrix system. The conventional model of a state-space system, which includes the dynamics and measurement equations, is augmented by an additional equation that models a deterministic linear combination of the noise-free K matrices $K_{k+1/k}^o$ and δK_{k+1}^o at time t_{k+1}. As will be shown in the sequel, this combination yields another noise-free K matrix $K_{k+1/k+1}^o$, which is the K matrix related to the true attitude at t_{k+1} and is computed using all the virtual noise-free vector measurements up to t_{k+1}. (We refer to these vectors as imaginary vectors because in reality only the noisy measurements are available to us.) Note that the structure of the computation of $K_{k+1/k+1}^o$ fits the structure of the update stage equation in REQUEST [Eq. (14)]. The reason for this computation will become clear when defining the estimation errors and deriving their recursive equations. As will be shown later, the updated estimation error is used to define a special cost function. This cost function will be minimized with respect to the scalar gain ρ_k, yielding the sought optimal gain ρ_k^*. This gain will be used in the update stage of the K-matrix estimation process, and, finally, an estimate of the attitude quaternion will be computed as the eigenvector that belongs to the largest eigenvalue of the updated K-matrix estimate.

Measurement Equation

The measurement equation of the K matrix is derived as follows: Consider the measurement equation for a single vector observation at time t_{k+1}, and assume that the reference unit vector $\boldsymbol{r}_{k+1}$ is known exactly. Denoting by A_{k+1} the attitude matrix at t_{k+1} and by $\boldsymbol{\delta b}_{k+1}$ the noise vector that corrupts the measurement $\boldsymbol{b}_{k+1}$, we have

$$\boldsymbol{b}_{k+1} = A_{k+1}\boldsymbol{r}_{k+1} + \boldsymbol{\delta b}_{k+1} \tag{15}$$

Define the following quantities:

$$B_b \triangleq a_{k+1}\boldsymbol{\delta b}_{k+1}\boldsymbol{r}_{k+1}^T, \qquad \mathcal{S}_b \triangleq {B_b}^T + B_b$$
$$z_b \triangleq a_{k+1}\boldsymbol{\delta b}_{k+1} \times \boldsymbol{r}_{k+1}, \qquad \kappa_b \triangleq tr(B_b) \tag{16}$$

Then, using these quantities, we can define a 4×4 symmetric matrix, denoted by V_{k+1}, as follows:

$$V_{k+1} = \frac{1}{a_{k+1}}\begin{bmatrix} \mathcal{S}_b - \kappa_b \boldsymbol{I}_3 & z_b \\ z_b^T & \kappa_b \end{bmatrix} \tag{17}$$

The matrix V_{k+1} is the error in the δK measurement [see Eqs. (12) and (13)] at t_{k+1} because

$$\delta K_{k+1} = \delta K_{k+1}^o + V_{k+1} \tag{18}$$

where, as mentioned before, δK_{k+1} and δK_{k+1}^o are, respectively, the noisy and the noise-free K matrices of the new vector available at t_{k+1}. Thus, δK_{k+1} is computed using the noisy observation, $\boldsymbol{b}_{k+1}, \boldsymbol{r}_{k+1}$ [see Eqs. (12) and (13)], while δK_{k+1}^o contains the imaginary noise-free vector measurements at t_{k+1}; that is, δK_{k+1}^o is expressed using Eqs. (12) and (13) for the noise-free observations (i.e., $\boldsymbol{\delta b} = \boldsymbol{0}$). Note that V_{k+1} is a linear function of $\boldsymbol{\delta b}_{k+1}$ and $\boldsymbol{r}_{k+1}$, and, because $\boldsymbol{\delta b}_{k+1}$ is random, V_{k+1} is also random. The linearity of V_{k+1} in $\boldsymbol{\delta b}_{k+1}$ will be used to derive a stochastic model for V_{k+1} from the stochastic model of $\boldsymbol{\delta b}_{k+1}$.

Process Equation

Let us denote by $\mathcal{B}_k$ and $\mathcal{R}$, respectively, the body frame at time t_k and the constant reference frame. Let the 3×1 vector $\boldsymbol{\omega}_k^o$ denote the body axes angular rate vector of $\mathcal{B}_k$ with respect to $\mathcal{R}$ expressed in $\mathcal{B}_k$, and let $K_{0/0}^o$ denote the noise-free (true) K matrix at time t_0. If $K_{0/0}^o$ and $\boldsymbol{\omega}_k^o$ are known, the true K matrix can be propagated using

$$K_{k+1/k}^o = \Phi_k^o K_{k/k}^o \Phi_k^{oT} \tag{19}$$

where $K_{k/k}^o$ is related to the true attitude at t_k and contains all of the noise-free observations up to t_k, $K_{k+1/k}^o$ is related to the true attitude at t_{k+1} and contains all of the noise-free observations up to t_k, and Φ_k^o is the 4×4 transition matrix corresponding to $\boldsymbol{\omega}_k^o$ [see Eqs. (9) and (10)]. In practice $\boldsymbol{\omega}_k^o$ is not known but is, rather, measured (or estimated). The transition matrix Φ_k, which is computed from the measured angular rate $\boldsymbol{\omega}_k$, contains an error term $\Delta\Phi_k$, that is,

$$\Phi_k^o = \Phi_k - \Delta\Phi_k \tag{20}$$

Using Eq. (20) in Eq. (19) yields

$$K_{k+1/k}^o = \Phi_k K_{k/k}^o \Phi_k^T + W_k^o \tag{21}$$

where

$$W_k^o \triangleq -\Phi_k K_{k/k}^o \Delta\Phi_k^T - \Delta\Phi_k^T K_{k/k}^o \Phi_k^T + \Delta\Phi_k K_{k/k}^o \Delta\Phi_k^T \tag{22}$$

The expression for W_k^o in Eq. (22) is exact but not useful because we cannot infer from it any statistical property of W_k^o. To determine a useful approximation to W_k^o, we assume that the error in $\boldsymbol{\omega}_k$ is additive. Denoting this error by $\boldsymbol{\epsilon}_k$, we have

$$\boldsymbol{\omega}_k = \boldsymbol{\omega}_k^o + \boldsymbol{\epsilon}_k \tag{23}$$

Based on this model we expand the expression for W_k^o in a Taylor series about $\boldsymbol{\omega}_k^o$, and, assuming that $\|\boldsymbol{\epsilon}_k\|$ is small enough, retain only the first-order term in $\boldsymbol{\epsilon}_k$. Let ${B^o}_{k/k}$ denote the true B matrix at time t_k [see Eq. (6)], that is, ${B^o}_{k/k}$ corresponds to the error-free vector observations up to t_k. Defining B_ϵ, S_ϵ, κ_ϵ, and z_ϵ as

$$B_\epsilon \triangleq [\boldsymbol{\epsilon}_k \times]B_{k/k}^o, \qquad S_\epsilon \triangleq B_\epsilon + B_\epsilon^T$$
$$[z_\epsilon \times] \triangleq B_\epsilon^T - B_\epsilon, \qquad \kappa_\epsilon \triangleq tr(B_\epsilon) \tag{24}$$

the first-order approximation in $\boldsymbol{\epsilon}_k$ for W_k^o is

$$W_k = \begin{bmatrix} S_\epsilon - \kappa_\epsilon \boldsymbol{I}_3 & z_\epsilon \\ z_\epsilon^T & \kappa_\epsilon \end{bmatrix}\Delta t \tag{25}$$

where $\Delta t = t_{k+1} - t_k$. The proof of Eq. (25) is detailed in Ref. 27 (pp. 233–235). To summarize, we have derived an approximate propagation equation of the true K matrix, which is

$$K_{k+1/k}^o = \Phi_k K_{k/k}^o \Phi_k^T + W_k \tag{26}$$

where Φ_k corresponds to $\boldsymbol{\omega}_k$ and W_k is an additive matrix noise. The initial true K matrix is denoted by $K_{0/0}^o$. The expression for W_k is provided in Eqs. (24) and (25). From these equations it is realized that W_k contains the noise vector $\boldsymbol{\epsilon}_k$ as well as all of the noise-free vector pairs $(\boldsymbol{b}^o, \boldsymbol{r})$ up to t_k. Note that W_k is random because $\boldsymbol{\epsilon}_k$ is random. Also note that W_k is linear in the error $\boldsymbol{\epsilon}_k$. This property will become useful when deriving a stochastic model for W_k. Extensive simulations show that the propagation model described by Eq. (26) is a good approximation to the exact propagation model described by Eq. (21) when W_k is used instead of W_k^o.

Pseudoprocess Equation

As mentioned earlier, a supplementary equation is added to the K-matrix model given in Eqs. (18) and (26). Motivated by the REQUEST algorithm [see Eq. (14)], we define a pseudoprocess as follows:

$$L_{k+1}^o \triangleq (1 - \alpha_{k+1})(m_k/m_{k+1})K_{k+1/k}^o + \alpha_{k+1}(\delta m_{k+1}/m_{k+1})\delta K_{k+1}^o \tag{27}$$

where α_{k+1} is any real number in the interval $[0, 1)$, the scalars m_{k+1} are recursively computed by

$$m_{k+1} = (1 - \alpha_{k+1})m_k + \alpha_{k+1}\delta m_{k+1}, \qquad m_0 = \delta m_0 \tag{28}$$

for $k = 0, 1 \ldots$, and δm_{k+1} is the positive weight assigned to δK_{k+1}, the pseudo K measurement at time $k+1$. Because the pseudo K measurement is constructed from a single $\boldsymbol{b}$ measurement, we choose $\delta m_{k+1} = a_{k+1}$. Note that Eq. (27), which we call pseudoprocess equation, has a structure that is similar to that of the update stage of REQUEST, given in Eq. (14). The pseudoprocess equation will be central in the development of Optimal-REQUEST. Next we show that L_{k+1}^o, an element of the pseudoprocess defined in Eq. (27), is a legitimate K matrix.

Proposition: For any value of α_{k+1} in the interval $[0, 1)$, the matrix L_{k+1}^o given in Eq. (27) is a K matrix related to the true attitude at time t_{k+1}.

Proof: By assumption, the two matrices $K_{k+1/k}^o$ and δK_{k+1}^o contain error-free vector observations; therefore, one of their eigenvectors is the true attitude quaternion $\boldsymbol{q}_{k+1}$, and it belongs to their maximal eigenvalue, which can be made equal to one by proper scaling. Thus,

$$K_{k+1/k}^o \boldsymbol{q}_{k+1} = \boldsymbol{q}_{k+1} \tag{29}$$

$$\delta K_{k+1}^o \boldsymbol{q}_{k+1} = \boldsymbol{q}_{k+1} \tag{30}$$

Using Eqs. (27–30), we find that

$$L^o_{k+1}\boldsymbol{q}_{k+1} = \left[(1-\alpha_{k+1})(m_k/m_{k+1})K^o_{k+1/k} + \alpha_{k+1}(\delta m_{k+1}/m_{k+1})\delta K^o_{k+1}\right]\boldsymbol{q}_{k+1}$$
$$= (1-\alpha_{k+1})(m_k/m_{k+1})\boldsymbol{q}_{k+1} + \alpha_{k+1}(\delta m_{k+1}/m_{k+1})\boldsymbol{q}_{k+1}$$
$$= \{[(1-\alpha_{k+1})m_k + \alpha_{k+1}\delta m_{k+1}]/m_{k+1}\}\boldsymbol{q}_{k+1} = \boldsymbol{q}_{k+1} \quad (31)$$

From Eq. (31) we conclude that $\boldsymbol{q}_{k+1}$ is an eigenvector of L^o_{k+1} associated with the eigenvalue 1. Consequently, the matrix L^o_{k+1} is a legitimate K matrix. □

As stated in the proposition, the matrix L^o_{k+1} is related to the true attitude at time t_{k+1}; this matrix is indeed a correct K matrix at time t_{k+1} containing all of the noise-free vector measurements up to time t_{k+1}. Adopting a consistent notation, we denote this matrix by $K^o_{k+1/k+1}$. The pseudoprocess equation is thus rewritten as

$$K^o_{k+1/k+1} = (1-\alpha_{k+1})(m_k/m_{k+1})K^o_{k+1/k} + \alpha_{k+1}(\delta m_{k+1}/m_{k+1})\delta K^o_{k+1} \quad (32)$$

It should be emphasized that Eq. (32) was developed to match the structure of the update stage of REQUEST. As will be seen later, this matching is crucial for defining the estimation errors, designing a cost function and, finally, minimizing this cost function with respect to the gain. The pseudoprocess of Eq. (32), and the process and measurement equations [Eqs. (26) and (18)], which are

$$K^o_{k+1/k} = \Phi_k K^o_{k/k}\Phi_k^T + W_k \quad (33)$$

$$\delta K_{k+1} = \delta K^o_{k+1} + V_{k+1} \quad (34)$$

(with an initial condition $K^o_{0/0}$), form the model for the K-matrix system.

Stochastic Models

The purpose of this subsection is to describe the stochastic models of the system noise matrices W_k and V_k. To derive the stochastic models for W_k and V_k, we need the stochastic models for $\boldsymbol{\epsilon}_k$ and $\boldsymbol{\delta b}_k$. As mentioned earlier, only basic models are considered in this work; thus, the vector process $\boldsymbol{\epsilon}_k$ is modeled as a zero-mean white-noise vector process whose components are identically distributed with variance η_k, that is,

$$E[\boldsymbol{\epsilon}_k] = 0, \qquad E\left[\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}^T_{k+i}\right] = \eta_k I_3 \delta_{k,k+i} \quad (35)$$

for $k = 1, 2, \ldots,$ and $\delta_{k,k+i}$ is the Kronecker delta function. Assuming that the unit vector measurements $\boldsymbol{b}_k$ are axisymmetrically distributed about their true value, we employ a unit vector error model[19] that provides approximate but quite accurate expressions for the mean and the covariance of $\boldsymbol{\delta b}_k$. The first and second moments of this model are

$$E[\boldsymbol{\delta b}_k] = 0, \qquad E\left[\boldsymbol{\delta b}_k\boldsymbol{\delta b}^T_{k+i}\right] = \mu_k\left(I_3 - \boldsymbol{b}_k\boldsymbol{b}^T_{k+i}\right)\delta_{k,k+i} \quad (36)$$

for $k = 1, 2, \ldots,$ where μ_k is the variance of the component of $\boldsymbol{b}_k$ along a direction normal to $E[\boldsymbol{b}_k]$. Furthermore, it is assumed that $\boldsymbol{\delta b}_k$ and $\boldsymbol{\epsilon}_k$ are mutually uncorrelated. The exact expressions for the two moments can be found in the appendix of Ref. 11.

Measure of Uncertainty

The goal of the following analysis is to present a model for the uncertainty that W_k and V_k introduce into the K-matrix system of Eqs. (33) and (34), respectively. For scalar and vector processes, the covariance is a measure of the uncertainty associated with the process error. In the case of a matrix process, like the one presented in Eqs. (33) and (34), we introduce a special measure of uncertainty as follows.

Definition 1: For the zero-mean general matrix process X, a measure of uncertainty P_{XX} is defined as

$$P_{XX} \triangleq E[XX^T] \quad (37)$$

As an example, consider the real 2×2 symmetric matrix of zero-mean processes x_{11}, x_{12}, and x_{22}, given by

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \quad (38)$$

The corresponding P_{XX} matrix has the following form:

$$P_{XX} = \begin{bmatrix} E\left[x_{11}^2\right] + E\left[x_{12}^2\right] & E[x_{11}x_{12}] + E[x_{12}x_{22}] \\ E[x_{11}x_{12}] + E[x_{12}x_{22}] & E\left[x_{12}^2\right] + E\left[x_{22}^2\right] \end{bmatrix} \quad (39)$$

Discussion

P_{XX} is a symmetric positive semidefinite matrix. The variances of each element of X are on the main diagonal of P_{XX}, whereas the off-diagonal elements of P_{XX} contain only cross-covariance terms.

Consider the 4×1 zero-mean vector vec$X$, defined as

$$\mathrm{vec}X \triangleq [x_{11} \quad x_{12} \quad x_{12} \quad x_{22}]^T \quad (40)$$

and construct its 4×4 covariance matrix

$$\mathrm{cov}(\mathrm{vec}X) \triangleq E[\mathrm{vec}X\,(\mathrm{vec}X)^T] \quad (41)$$

Then, the trace of the matrix P_{XX} is identical to the trace of the covariance matrix in Eq. (41), that is

$$tr(P_{XX}) \equiv tr[\mathrm{cov}(\mathrm{vec}X)] \quad (42)$$

We realize that although the matrix P_{XX} is not a covariance matrix, it has a desired feature in a compact convenient form. Thus, it will be used as a measure of uncertainty for the matrix error processes that are considered in this paper.

Using the expressions for W_k and V_k in Eqs. (25) and (17), respectively, it can be shown that these processes are zero-mean uncorrelated white-noise processes. Thus,

$$E\left[W_kW^T_{k+i}\right] = O_4, \qquad E\left[V_kV^T_{k+i}\right] = O_4, \qquad E\left[W_kV^T_{k+i}\right] = O_4 \quad (43)$$

for $i \neq 0$. Denote by $\mathcal{Q}_k$ and $\mathcal{R}_k$, respectively, the P matrices of W_k and V_k, that is,

$$\mathcal{Q}_k \triangleq E\left[W_kW_k^T\right], \qquad \mathcal{R}_k \triangleq E\left[V_kV_k^T\right] \quad (44)$$

then, explicit expressions for $\mathcal{Q}_k$ and $\mathcal{R}_k$ can be derived using the assumptions we made on $\boldsymbol{\epsilon}_k$ and $\boldsymbol{\delta b}_k$. In the computation of $\mathcal{Q}_k$, one must address the issue of the dependence of the process noise matrix W_k on the noise-free matrix $B^o{}_{k/k}$ [Eqs. (24) and (25)], which is unknown. To overcome this difficulty, $B^o{}_{k/k}$ is replaced by its best available estimate $B_{k/k}$. The latter is computed from the estimated K matrix $K_{k/k}$ using the definition of the K matrix given in Eqs. (6) and (7). The detailed computation of $\mathcal{Q}_k$ and $\mathcal{R}_k$ is provided in Ref. 27 (pp. 235–240).

Optimal-REQUEST

The algorithm derivation in this section follows an approach that was used in a direct derivation of the Kalman–Bucy filter for the case of vector processes.[28] A similar derivation of the discrete Kalman filter can be found in the literature (e.g., see Ref. 26, pp. 105–110). The approach consists of three steps:

1) The update of the estimate is formulated as a linear combination of the predicted estimate and the new observation.

2) The a posteriori and a priori estimates are forced to be unbiased.

3) The optimal filter gain is computed by minimizing the variance of the a posteriori estimation error.

In the third step, instead of the variance, we will use the proxy measure of uncertainty introduced in the preceding section.

Measurement Update Stage

The proposed update stage is a slightly modified version of REQUEST, namely, unlike the REQUEST update formula of Eq. (14), here the updated estimate $K_{k+1/k+1}$ is chosen to be a convex combination of the predicted estimate $K_{k+1/k}$ and the new observation δK_{k+1}, that is,

$$K_{k+1/k+1} = (1-\rho_{k+1})(m_k/m_{k+1})K_{k+1/k} + \rho_{k+1}(\delta m_{k+1}/m_{k+1})\delta K_{k+1} \tag{45}$$

where δm_{k+1} is a positive scalar weight and m_{k+1} is computed recursively by

$$m_{k+1} = (1-\rho_{k+1})m_k + \rho_{k+1}\delta m_{k+1} \tag{46}$$

for $k = 0, 1 \ldots$, and $m_0 = \delta m_0$. The scalar $\rho_{k+1} \in [0, 1)$ has the role of a gain in Eq. (45). (We exclude the case where $\rho_{k+1} = 1$ for practical reasons because, in that case, the filter would be memoryless.) Because of the linearity of Eq. (45) and the fact that ρ_{k+1} is a scalar, the properties of symmetry and of zero trace of the estimated K matrix are preserved. This is a matter of importance because the computation of the attitude quaternion, which uses the $\boldsymbol{q}$ method, depends on these properties.

The estimation errors of the filter are defined by

$$\Delta K_{k+1/k} \triangleq K^o_{k+1/k} - K_{k+1/k} \tag{47a}$$

$$\Delta K_{k+1/k+1} \triangleq K^o_{k+1/k+1} - K_{k+1/k+1} \tag{47b}$$

where $\Delta K_{k+1/k}$ and $\Delta K_{k+1/k+1}$ denote, respectively, the a priori and the a posteriori estimation errors. It is assumed that the a priori estimate $K_{k+1/k}$ is unbiased, that is, $E[\Delta K_{k+1/k}] = 0$. This assumption will be justified when developing the prediction stage.

The expression for $K^o_{k+1/k+1}$ is known from Eq. (32), where α_{k+1} is arbitrary in [0, 1). For a reason that will become clear in the ensuing, we choose $\alpha_{k+1} = \rho_{k+1}$, where ρ_{k+1} is that which was introduced in Eq. (45). Consequently, Eq. (32) becomes

$$K^o_{k+1/k+1} = (1-\rho_{k+1})(m_k/m_{k+1})K^o_{k+1/k} + \rho_{k+1}(\delta m_{k+1}/m_{k+1})\delta K^o_{k+1} \tag{48}$$

Subtracting Eq. (45) from Eq. (48) and making use of the definitions in Eqs. (47) yields the following relation between the a priori and the a posteriori errors:

$$\Delta K_{k+1/k+1} = (1-\rho_{k+1})(m_k/m_{k+1})\Delta K_{k+1/k} + \rho_{k+1}(\delta m_{k+1}/m_{k+1})V_{k+1} \tag{49}$$

where V_{k+1} is the measurement error defined in Eq. (17). Note that the expression in Eq. (49) is based on the choice $\alpha_{k+1} = \rho_{k+1}$. Taking the expectation of both sides of Eq. (49) yields

$$E[\Delta K_{k+1/k+1}] = (1-\rho_{k+1})(m_k/m_{k+1})E[\Delta K_{k+1/k}] + \rho_{k+1}(\delta m_{k+1}/m_{k+1})E[V_{k+1}] \tag{50}$$

Using the assumptions that the measurement error V_{k+1} and the a priori estimation error $\Delta K_{k+1/k}$ are zero mean, one finds that the a posteriori estimation error $\Delta K_{k+1/k+1}$ is zero mean too, as required.

The P matrices corresponding to both estimation errors are defined as follows:

$$P_{k+1/k} \triangleq E\left[\Delta K_{k+1/k}\Delta K^T_{k+1/k}\right] \tag{51}$$

$$P_{k+1/k+1} \triangleq E\left[\Delta K_{k+1/k+1}\Delta K^T_{k+1/k+1}\right] \tag{52}$$

Using Eq. (49), we compute the following product:

$$\begin{aligned}\Delta K_{k+1/k+1}\Delta K^T_{k+1/k+1} &= [(1-\rho_{k+1})(m_k/m_{k+1})]^2 \\ &\times \Delta K_{k+1/k}\Delta K^T_{k+1/k} - (1-\rho_{k+1})\rho_{k+1}\left(m_k\delta m_{k+1}\big/m^2_{k+1}\right) \\ &\times \left(\Delta K_{k+1/k}V^T_{k+1} + V_{k+1}\Delta K^T_{k+1/k}\right) \\ &+ [\rho_{k+1}(\delta m_{k+1}/m_{k+1})]^2 V_{k+1}V^T_{k+1}\end{aligned} \tag{53}$$

Before computing the expectations of the expressions in Eq. (53), we consider the following expectation, $E[\Delta K_{k+1/k}V^T_{k+1}]$. From Eq. (36) $\boldsymbol{\delta b}$ is a zero-mean white-noise process; therefore [see Eqs. (15–17)], V is a zero-mean white-noise process too. Moreover, the random variable $\Delta K_{k+1/k}$ depends on the sequences $\{W_i\}$ and $\{V_i\}$, where i takes the integer values from 1 to k only. Therefore $\Delta K_{k+1}/k$ and V_{k+1} are uncorrelated; thus,

$$E\left[\Delta K_{k+1/k}V^T_{k+1}\right] = O_4 \tag{54}$$

Taking the transpose of Eq. (54) yields a similar result. Taking the expectation of both sides of Eq. (53), and only retaining the nonzero terms, yields

$$P_{k+1/k+1} = [(1-\rho_{k+1})(m_k/m_{k+1})]^2 E\left[\Delta K_{k+1/k}\Delta K^T_{k+1/k}\right] + [\rho_{k+1}(\delta m_{k+1}/m_{k+1})]^2 E\left[V_{k+1}V^T_{k+1}\right] \tag{55}$$

One identifies the matrices $P_{k+1/k}$ and $\mathcal{R}_{k+1}$ in the first and second terms of the right-hand side of Eq. (55); thus, we can write

$$P_{k+1/k+1} = [(1-\rho_{k+1})(m_k/m_{k+1})]^2 P_{k+1/k} + [\rho_{k+1}(\delta m_{k+1}/m_{k+1})]^2\mathcal{R}_{k+1} \tag{56}$$

Equation (56) expresses the uncertainty update in the K-matrix estimation process for any ρ_{k+1}, when a new measurement is acquired.

Optimal Gain

When a new observation is processed, we would like the estimation uncertainty to decrease as much as possible. From the earlier discussion of the properties of the P matrix, we saw that its trace was a suitable measure of the uncertainty. Thus, we define the following cost function:

$$J_{k+1} \triangleq tr\left(E\left[\Delta K_{k+1/k+1}\Delta K^T_{k+1/k+1}\right]\right) = tr(P_{k+1/k+1}) \tag{57}$$

Then, the design problem of the filter gain ρ_{k+1} reduces to solving the following minimization problem with respect to ρ_{k+1}:

$$\min_{\rho_{k+1}\in[0,1)} J_{k+1} \tag{58a}$$

where

$$J_{k+1} \triangleq tr(P_{k+1/k+1}) \tag{58b}$$

Inserting Eq. (56) into the expression for J_{k+1} in Eq. (57) yields, after some manipulation,

$$J_{k+1}(\rho_{k+1}) = [(1-\rho_{k+1})(m_k/m_{k+1})]^2 tr(P_{k+1/k}) + [\rho_{k+1}(\delta m_{k+1}/m_{k+1})]^2 tr(\mathcal{R}_{k+1}) \tag{59}$$

The first-order necessary condition for an extremum of J_{k+1} is

$$\frac{\mathrm{d}J_{k+1}}{\mathrm{d}\rho_{k+1}} = 2\left[\left(\frac{m_k}{m_{k+1}}\right)^2 tr(P_{k+1/k}) + \left(\frac{\delta m_{k+1}}{m_{k+1}}\right)^2 tr(\mathcal{R}_{k+1})\right]\rho_{k+1} - 2\left(\frac{m_k}{m_{k+1}}\right)^2 tr(P_{k+1/k}) = 0 \tag{60}$$

resulting in the following condition for ρ^*_{k+1} to yield a stationary point for J_{k+1}

$$\rho^*_{k+1} = \frac{m_k^2 tr(P_{k+1/k})}{m_k^2 tr(P_{k+1/k}) + \delta m^2_{k+1} tr(\mathcal{R}_{k+1})} \tag{61}$$

Using the sufficiency condition for this point to be a minimum, it can be verified that the cost function J_{k+1} indeed reaches a minimum at ρ^*_{k+1}. Note that, in the case of a scalar process, Eq. (61) yields the exact expression for the Kalman-filter gain. Even in the general case the filter gain ρ^*_{k+1} still has the feature of a Kalman-filter gain; namely, for a high uncertainty in the a priori estimate, relative to the uncertainty in the measurement, the gain is close to 1 and weighs more favorably the new measurement in the update stage described in Eq. (45). On the other hand, for a high uncertainty in the measurement, relative to the uncertainty in the a priori estimate, the gain is close to 0, and the filter assigns a low weight to the new measurement.

Prediction Stage

We require that the predicted $K_{k+1/k}$ be linear in $K_{k/k}$ and that it produces an unbiased predicted estimate. These requirements yield the prediction formula of the REQUEST algorithm [Eq. (11)]

$$K_{k+1/k} = \Phi_k K_{k/k} \Phi_k^T \tag{62}$$

Using the K-matrix process of Eq. (26), the prediction in Eq. (62), and the definitions of the estimation errors given in Eqs. (47) yields the error propagation equation

$$\Delta K_{k+1/k} = \Phi_k \Delta K_{k/k} \Phi_k^T + W_k \tag{63}$$

It is easy to see that $E[\Delta K_{k+1/k}] = 0$, which justifies our initial assumption that the a priori estimate of $K_{k+1/k}$ is unbiased. Using the latter expression for $\Delta K_{k+1/k}$, the stochastic models of the noise, and the orthogonality property of the Φ_k matrix, the propagation equation for the P matrix is obtained as follows:

$$P_{k+1/k} = \Phi_k P_{k/k} \Phi_k^T + \mathcal{Q}_k \tag{64}$$

Equation (64) is similar to the covariance propagation equation in the Kalman-filter algorithm.

To conclude, similarly to the Kalman filter, the algorithm comprises two parallel channels. One channel is for the computation of the signal estimate, which here is the K-matrix estimate, and the other one is for the uncertainty propagation of the estimation process, which in the Kalman filter is expressed by the covariance matrix, and is needed for the computation of the optimal filter gain.

Algorithm Summary

The Optimal-REQUEST algorithm presented in this section can be summarized as follows:

Initialization:

$$K_{0/0} = \delta K_0 \tag{65}$$

where δK_0 is computed using the first vector measurement according to Eqs. (12) and (13).

$$P_{0/0} = \mathcal{R}_0 \tag{66}$$

$$m_0 = \delta m_0 \tag{67}$$

where δm_0 is a positive weighting factor.

Time update:

$$K_{k+1/k} = \Phi_k K_{k/k} \Phi_k^T \tag{68}$$

$$P_{k+1/k} = \Phi_k P_{k/k} \Phi_k^T + \mathcal{Q}_k \tag{69}$$

where the matrix $\mathcal{Q}_k$ is computed according to Eqs. (24), (25), and (44).

Measurement update:

$$\rho_{k+1}^* = \frac{m_k^2 tr(P_{k+1/k})}{m_k^2 tr(P_{k+1/k}) + \delta m_{k+1}^2 tr(\mathcal{R}_{k+1})} \tag{70}$$

$$m_{k+1} = \left(1 - \rho_{k+1}^*\right) m_k + \rho_{k+1}^* \delta m_{k+1} \tag{71}$$

$$K_{k+1/k+1} = \left(1 - \rho_{k+1}^*\right) \frac{m_k}{m_{k+1}} K_{k+1/k} + \rho_{k+1}^* \frac{\delta m_{k+1}}{m_{k+1}} \delta K_{k+1} \tag{72}$$

$$P_{k+1/k+1} = \left[\left(1 - \rho_{k+1}^*\right) \frac{m_k}{m_{k+1}}\right]^2 P_{k+1/k} + \left(\rho_{k+1}^* \frac{\delta m_{k+1}}{m_{k+1}}\right)^2 \mathcal{R}_{k+1} \tag{73}$$

where the matrix $\mathcal{R}_{k+1}$ is computed according to Eqs. (16), (17), and (44). The optimal quaternion $\hat{\boldsymbol{q}}_{k+1/k+1}$ is the eigenvector of $K_{k+1/k+1}$, which belongs to its maximal eigenvalue.

Simulation Study

An extensive Monte Carlo (MC) simulation study was performed in order to test the attitude estimator in the presence of process and measurement noises. Different single vector observations were acquired at each sampling time. The body coordinate system $\mathcal{B}$ was assumed to be fixed with respect to an inertial coordinate system $\mathcal{R}$. The vector observation noise was a zero-mean Gaussian white noise with an angular standard deviation of one degree, which is a typical accuracy obtained using magnetometers. Three body-mounted gyroscopes measured the angular velocity of $\mathcal{B}$ with respect to $\mathcal{R}$. Because the system $\mathcal{B}$ did not rotate with respect to system $\mathcal{R}$, the nominal body rates were zero; hence, the gyro measurements contained only gyro noises. The gyro noises were Gaussian zero-mean white noises with a standard deviation of 0.2 deg/h in each axis. The noise models in the system and in the filter were identical. The sample rate was 10 Hz, both in the measured directions and in the gyro measurements. Each simulation lasted 6000 s. We ran 100 runs with different seeds and averaged the results at each time point to obtain the ensemble averages.

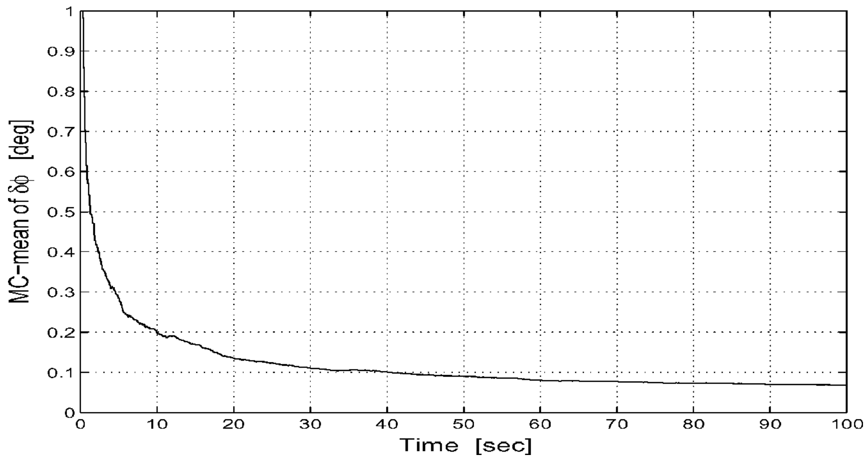
The results are summarized in Figs. 1 and 2 and Table 1. Figure 1a presents the time history of $\overline{\delta\phi}$, the MC mean of $\delta\phi$, the angular estimation error. The angle $\delta\phi$ is defined as the angle of the small rotation that brings the estimated body frame $\hat{\mathcal{B}}$ onto the true body frame $\mathcal{B}$. This angle is obtained as follows: First, the quaternion of the rotation from $\hat{\mathcal{B}}$ to $\mathcal{B}$, denoted by $\boldsymbol{\delta q}$, is evaluated, then the rotation angle $\delta\phi$ is computed from δq, the scalar component of $\boldsymbol{\delta q}$, using the known relation (Ref. 1, p. 414) $\delta\phi = 2\arccos(\delta q)$. The error reaches a steady state of approximately 0.06 deg. Figure 1b shows the time history of the MC mean of ρ^*, the optimal filter gain. The average gain decreases exponentially from 0.5 down to 0.001. As expected, the Optimal-REQUEST algorithm behaves like a Kalman filter; it initially weighs more the new observations in the estimate, and, as the number of processed measurements grows, it progressively turns to be a pure predictor of the estimate, that is, it weighs less the incoming measurements. The attitude estimated by the Optimal-REQUEST filter converges successfully to the true attitude.

Figure 2 shows the performance of Optimal-REQUEST compared to that of various cases of REQUEST where the gain ρ is chosen as a constant. We chose three different values for ρ, namely, 0.1, 0.01, and 0.001, which, as seen in Fig. 1b are typical values in the span of the optimal gain ρ^*. In this simulation the sampling frequency F_s was chosen equal to 0.5 Hz. Figure 2 depicts the variations of the MC mean of $\delta\phi$ for each case. For $\rho = 0.1$, the filter weighs relatively much the measurements, so that after a quick transient the error remains on a relatively high steady state (0.45 deg) and shows random variations. On the other hand, a very low gain ($\rho = 0.001$) yields a smooth but very slow convergence of the error, which reaches the value of 0.05 deg in steady state. As seen in Fig. 2, the optimal gain ρ^* of Optimal-REQUEST yields a lower bound for the various MC means of $\delta\phi$. This is true during the transient as well as in the steady state where Optimal-REQUEST yields an error of 0.03 deg.

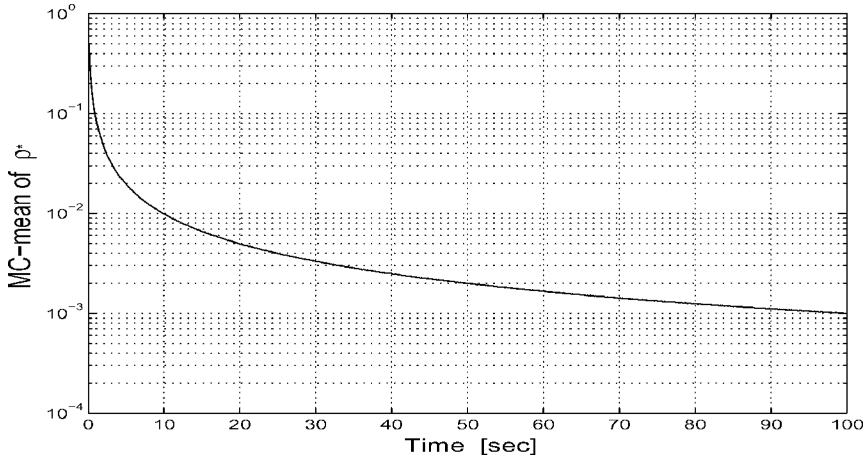
Optimal-REQUEST was also tested using various sampling rates F_s, standard deviations of the vector observations $\sqrt{\mu}$, and standard deviations of the gyro output $\sqrt{\eta}$. Table 1 presents the Monte Carlo means of $\delta\phi$ at the final time $\overline{\delta\phi}(t_f)$ for the various cases. For the sake of comparison, the same number of observations—here, 2000—are

Table 1 Monte Carlo means of $\delta\phi$ at final time

F_s, Hz	$\sqrt{\mu}$, deg	$\overline{\delta\phi}$ at t_f $\sqrt{\eta} = 0.01$, deg/h	$\overline{\delta\phi}$ at t_f $\sqrt{\eta} = 360$, deg/h	$\overline{\delta\phi}$ at t_f $\sqrt{\eta} = 3600$, deg/h
10	1	0.03	0.15	0.78
10	5	0.15	0.55	1.99
0.5	1	0.04	0.39	3.25
0.5	5	0.24	1.18	7.79



a) Time history of the MC-mean of $\delta\phi$



b) Time history of the MC-mean of ρ^*

Fig. 1 Performance of Optimal-REQUEST averaged over 100-runs Monte Carlo simulation.

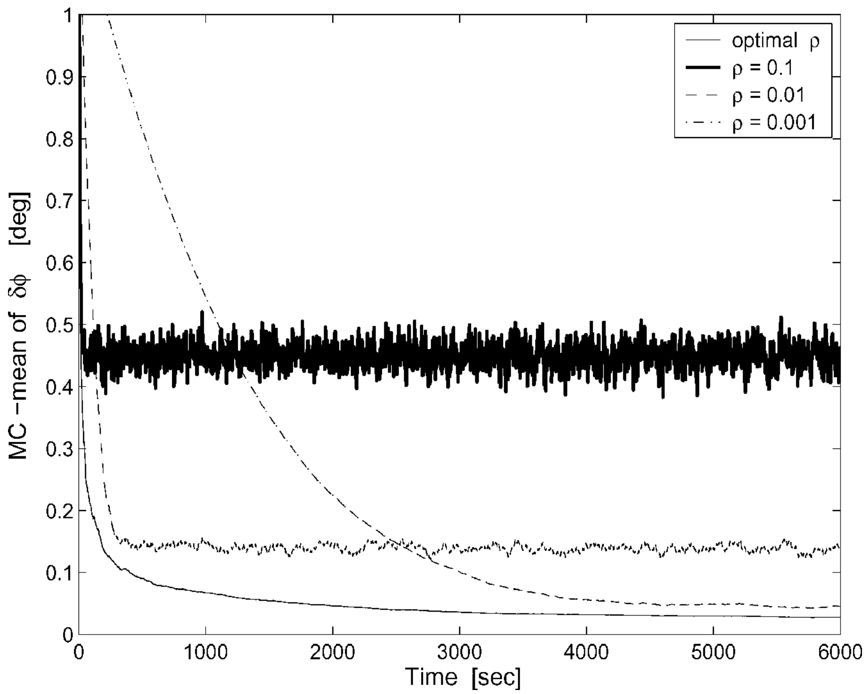


Fig. 2 Comparison of Optimal-REQUEST and REQUEST.

processed in each case. As seen in Table 1, the values of $\overline{\delta\phi}(t_f)$ are consistent with those of the filter parameters, that is, $\overline{\delta\phi}(t_f)$ increases with $1/F_s$, $\sqrt{\mu}$, and $\sqrt{\eta}$. Notice that $\overline{\delta\phi}(t_f)$ converges to some final value even in the worst simulated case.

Conclusions

In this paper a new recursive optimal estimator for estimating the quaternion of rotation from vector measurements is presented. Named Optimal-REQUEST, this algorithm is an extension of the REQUEST algorithm, which is based on the $\boldsymbol{q}$ method.

The proposed filter is derived using a unique variance-like performance criterion, which gives a stochastic basis to the estimation process. Like a Kalman filter, the proposed algorithm optimally filters both measurement and process noises; thus, it covers a deficiency of REQUEST, where the process noise is filtered in an empirical manner. The cost to achieve that performance is that the computation of the optimal gain involves 4×4 matrix equations, which increases the computational burden of the filter. The special case of a zero-mean white-noise process is considered here. When severe modeling errors, like unknown constant gyro biases, are present, adaptive filtering theory can be used to adapt online the covariance of the filter process noise to the bias level, rendering the overall estimation scheme robust to the modeling uncertainty.

Optimal-REQUEST also retains all of the features of REQUEST; that is, it is a time-varying recursive attitude quaternion estimator, the quaternion unit-norm property is preserved, and the attitude is updated even when a single vector observation is processed at each sampling time. The efficiency of the new filter is demonstrated through Monte Carlo simulations. A comparison test was performed on Optimal-REQUEST and several versions of REQUEST where, as the algorithm is set, for each version, the fading memory factor was held constant during the whole simulation period. The simulation results show that Optimal-REQUEST yields the lowest Monte Carlo mean of the angular estimation error during the transient period as well as at steady state, clearly demonstrating the superiority of the new algorithm over the old one.

Acknowledgments

This work was partially supported by NASA Grant NAG5-8770. We wish to thank T. Stengle and R. Harman of NASA Goddard Space Flight Center for this support. A part of this work was performed while the second author was holding a National Research Council Research Associateship Award at NASA Goddard Space Flight Center.

References

[1]Wertz, J. R. (ed.), *Spacecraft Attitude Determination and Control*, D. Reidel, Dordrecht, The Netherlands, 1984.

[2]Black, H. D., "A Passive System for Determining the Attitude of a Satellite," *AIAA Journal*, Vol. 2, No. 7, 1964, pp. 1350, 1351.

[3]Davenport, P., "Attitude Determination and Sensor Alignment via Weighted Least Squares Affine Transformations," NASA, X-514-71-312, Aug. 1971.

[4]Wahba, G., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 7, No. 3, 1965, p. 409.

[5]Farrel, J. L., and Stuelpnagel, J. C., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 8, No. 3, 1966, pp. 384–386.

[6]Wessner, R. H., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 8, No. 3, 1966, pp. 384–386.

[7]Velman, J. R., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 8, No. 3, 1966, pp. 384–386.

[8]Brock, J. E., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 8, No. 3, 1966, pp. 384–386.

[9]Brock, J. E., "Optimal Matrices Describing Linear Systems," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1292–1296.

[10]Markley, F. L., "Attitude Determination Using Vector Observations: a Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences*, Vol. 41, No. 2, 1993, pp. 261–280.

[11]Markley, F. L., "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *Journal of the Astronautical Sciences*, Vol. 36, No. 3, 1988, pp. 245–258.

[12]Bar-Itzhack, I. Y., "Polar Decomposition for Attitude Determination from Vector Observations," *Flight Mechanics/Estimation Theory Symposium*, NASA Conf. Pub. 3186, NASA, 1992, pp. 243–257.

[13]Bar-Itzhack, I. Y., and Markley, F. L., "Unconstrained Optimal Transformation Matrix," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 34, No. 1, 1998, pp. 338–340.

[14]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[15]Markley, F. L., "Spacecraft Attitude Determination Methods," *40th Israel Annual Conference on Aerospace Sciences*, edited by Y. Oshman, Vol. 40, Technion—Israel Inst. of Technology, Haifa, Israel, 2000.

[16]Keat, J., "Analysis of Least Squares Attitude Determination Routine DOAOP," Computer Sciences Corp., CSC/TM-77/6034, Silver Spring, MD, Feb. 1977.

[17]Bellman, R., *Introduction to Matrix Analysis*, McGraw–Hill, New-York, 1960, pp. 5, 6.

[18]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, 1983, p. 405.

[19]Shuster, M. D., and Oh, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, 1981, pp. 70–77.

[20]Mortari, D., "A Closed-Form Solution to the Wahba Problem," *Journal of the Astronautical Sciences*, Vol. 45, No. 2, 1997, pp. 195–204.

[21]Mortari, D., "Second Estimator of the Optimal Quaternion," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 5, 2000, pp. 885–888.

[22]Markley, F. L., "Attitude Determination Using Vector Observations: Theory," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, 1989, pp. 41–58.

[23]Shuster, M. D., "A Simple Kalman Filter and Smoother for Spacecraft Attitude," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, 1989, pp. 89–106.

[24]Bar-Itzhack, I. Y., "REQUEST: A Recursive QUEST Algorithm for Sequential Attitude Determination," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 5, 1996, pp. 1034–1038.

[25]Psiaki, M. L., "Attitude Determination Filtering via Extended Quaternion Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, 2000, pp. 206–214.

[26]Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1988.

[27]Choukroun, D., "Novel Methods for Attitude Estimation from Vector Observations," Ph.D. Dissertation, Faculty of Aerospace Engineering, Technion—Israel Inst. of Technology, Haifa, Israel, May 2003.

[28]Athans, M., and Tse, E., "A Direct Derivation of the Optimal Linear Filter Using the Maximum Principle," *IEEE Transactions on Automatic Control*, Vol. AC-12, No. 6, 1967, pp. 690–698.