**Application of a GLR detector/estimator to the terminal guidance problem of intercepting a maneuvring ballistic missile**

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## Symbols and Abbreviations

| | | |
|---|---|---|
| $a_{P,E}$ | : | achieved acceleration |
| $a_{P,E}^{c}$ | : | commanded acceleration |
| $a_{P,E}^{max}$ | : | maximal achievable acceleration |
| $A$ | : | continuous time state transition matrix |
| $B$ | : | continuous time input matrix |
| $d$ | : | correlation between the sequences of innovation and of signature |
| $F$ | : | discrete time state transition matrix |
| $G_{1,2}$ | : | discrete time input matrix |
| $g$ | : | maximum likelihood ratio |
| $h$ | : | detection threshold |
| $H$ | : | measurement matrix |
| $J$ | : | Kullback-Leibler divergence |
| $k$ | : | time |
| $k_0$ | : | reinitialization time |
| $k_s$ | : | switch time |
| $Kalman/SF$ | : | Kalman filter with shaping filter |
| $GLR$ | : | Generalized Likelihood Ratio |
| $l$ | : | likelihood ratio |
| $I$ | : | identity matrix |
| $M$ | : | state correction matrix of a mismatched dynamic profile |
| $P$ | : | state estimate covariance matrix |
| $Q$ | : | continuous time process noise covariance matrix |
| $Q_k$ | : | discrete time process noise covariance matrix |
| $R$ | : | range |
| $R_{E,P,1,2}$ | : | cartesian position |
| $S_k$ | : | innovation covariance matrix |
| $SSK$ | : | single shot kill |
| $u_1$ | : | known input |
| $u_2$ | : | unknown input |
| $U_2$ | : | true dynamical profile |
| $\bar{U}_2$ | : | normalized dynamical profile |
| $\hat{U}_2$ | : | estimated dynamical profile |
| $V_m$ | : | sequence of innovation from the mismatched filter (vector) |
| $v_{P,E}$ | : | velocity |
| $W$ | : | Kalman gain |
| $w$ | : | process noise |
| $x$ | : | state vector |
| $y$ | : | measurement |

| | | |
|---|---|---|
| $\alpha$ | : | false alarm rate |
| $\beta$ | : | evader heading angle |
| $\Delta$ | : | time interval between two measurements |
| $\Delta\nu$ | : | error on the estimated magnitude of a dynamic profile |
| $\hat{\Delta}U_2$ | : | dynamical profile mismatched for state estimation |
| $\eta$ | : | measurement noise |
| $\Gamma$ | : | state prediction of a dynamic profile |
| $\gamma$ | : | pursuer heading angle |
| $\nu$ | : | true magnitude of the dynamic profile |
| $\hat{\nu}$ | : | estimated dynamic profile magnitude |
| $\Phi$ | : | transition matrix applied to the preceding state prediction |
| $\rho$ | : | signature in the innovation |
| $\sigma^2$ | : | measurement noise covariance (angular) |
| $\sigma_y^2$ | : | measurement noise covariance (cartesian) |
| $\Sigma$ | : | bloc matrix build from the sequence of innovation covariance |
| $\tau_{P,E}$ | : | time constant lag |
| $\Upsilon$ | : | sequence of signature in the innovation (vector) |
| $\Xi$ | : | a component of the state prediction of a reinitialization error's dynamic profile |

## 1. Introduction

In the last few years, successful attempts to intercept a non-maneuvring ballistic missile have been realized. This is a significant achievement because the constraint on the homing accuracy to intercept a ballistic missile with an interceptor are considerable: a hit-to-kill is required for a successful interception. However, the interception of a maneuvring ballistic missile is still an open issue. The main problem stems from the imperfect information on the evader state, especially the uncertainty about the evader acceleration. In situations like tactical engagement, the interceptor may enjoy a large maneuvrability advantage over the evader, effectively allowing the interceptor to cope with a maneuvring evader despite imperfect information on the evader state. In ballistic engagement, the interceptor does not enjoy any longer a significant maneuvrability advantage over the evader. The accuracy of the homing loop is then very dependent on the evader maneuvers; in particular, abrupt changes in the evader acceleration can have a devastating effect on the homing accuracy. Quick detection of such changes is likely to improve the performance of the interceptor.

Various techniques has been developed to detect changes in dynamical systems. These techniques can be grouped into three main categories: shaping filter methods, multiple model filter methods and residual based methods. A review of these techniques is presented in Bar-Shalom *et al.* 2001 (with an emphasis on multiple model methods) and in Lai 1995 (residual based methods only). One of the residual based techniques of interest is the Generalized Likelihood Ratio (GLR) test. The GLR test can handled situations in which the magnitude of the change and the onset time of the change are both unknown. From this GLR test for detecting a change, it is possible to get an estimate of the magnitude of the change and an estimate of the change time. From a theoretical point of view, some optimality results related to the delay of detection have been demonstrated (Lai 2000) for the GLR test and other closely related likelihood ratio tests

A few studies about using a GLR detector/estimator in pursuit/evasion problems have been published, see Dowdle *et al.* 1982 and Korn *et al.* 1982. These studies addressed the tracking performance of the GLR detector/estimator but not the performance of the whole homing loop. Also, the examples presented in these papers are limited to the detection of a single change and perfect information on the initial state is provided. The aim of this work is to assess the performance of the homing loop using a GLR detector/estimator during the terminal guidance phase of an engagement between an anti-missile (the pursuer) and a maneuvring ballistic missile (the evader) and in situations in which the initial target acceleration is unknown. The GLR detector/estimator is described in section 2. The details of the simulation code for the pursuit/evasion engagement are exposed in section 3. The results obtained are presented and discussed in section 4. The section 5 contains the conclusion and some future avenues of research.

## 2. GLR detector/estimator

The GLR detector has been introduced by Willsky & Jones (1976) along with a variation of it suitable for practical implementation known as a window limited GLR.

For detection of a change, the detector described in this section closely follows the window limited GLR of Willsky & Jones except for the addition of a signature to probe for an eventual reinitialization error. This addition allows increasing the robustness of the detector.

For state estimation, the approach presented here is different from the approach of Willsky & Jones 1976. The state estimation filter is separated from the detection filter. The GLR detector is used to adapt online the bandwidth of the state estimation filter and to correct the state estimate. Before the detection of a change, the shaping filter has a high bandwidth. Upon detection of a switch, the state estimate is corrected using the maximum likelihood estimation of the unknown input dynamic profile generated by the GLR detector. The bandwidth of the shaping filter is also reduced for some period of time. This period of time should depend on the lower bound for the time interval between switches of dynamic profiles.

### 2.1. Problem definition

Let the system model be:

$$\begin{aligned} x(k+1) &= Fx(k) + G_1 u_1(k) + G_2 u_2(k) + w(k+1), && x \in \Re^n, u_{1,2} \in \Re^1, w \in \mathcal{N}(0, Q_w) \\ y(k) &= Hx(k) + \eta(k), && y \in \Re^m, \eta \in \mathcal{N}(0, Q_\eta) \end{aligned}$$

Here, $w$, $\eta$ are white gaussian noise, $u_1$ is a known input and $u_2$ is an unknown disturbance. Although $u_2(k)$ is unknown, the sequence $u_2(\cdot)$ is assumed to belong to a known finite set of dynamic profiles of unknown magnitude and starting at unknown time. Moreover, the time period of eventual switches between dynamic profiles by $u_2(\cdot)$ is bounded from below.

The problem is to estimate the system state online. To do so, the starting time of the change corresponding to the dynamic profile of $u_2(\cdot)$ is identified along with its magnitude using a Generalized Likelihood Ratio (GLR) approach. The GLR technique is a sequential probability ratio test to decide between the hypothesis $H_0$ "no change occurred" versus the hypothesis $H_1$ "a change occurred before time $k$". It applies to situations where the time of the switch and the magnitude of the change are a priori unknown and are to be estimated; the estimation of these parameters being done using a maximum likelihood technique.

### 2.2. Algorithm

The algorithm is working in four steps:

1. A Kalman filter is designed assuming without loss of generality, $u_2(\cdot) = 0$. This filter is denoted as the mismatched filter. If this assumption on $u_2(\cdot)$ is true, the generated innovation sequence $V_m$ is an independent sequence with a gaussian distribution and zero mean. If the assumption on $u_2(\cdot)$ is wrong, then the sequence $V_m$ is a gaussian distribution of zero mean with the addition of a sequence of signature $\Upsilon(u_2)$ (see Basseville & Nikiforov 1993, p. 240):

$$\mathcal{L}(V_m) = \begin{cases} \mathcal{N}(0, \Sigma), & u_2(i) = 0 \, \forall i \leq k \\ \mathcal{N}(0, \Sigma) + \Upsilon(u_2), & u_2(i) \neq 0 \, \exists i \leq k \end{cases}$$

   In order to get a large signature if the assumption on $u_2(\cdot)$ is wrong, and thus improves its detectability, the mismatched filter should have a low bandwidth.

2. The GLR test is applied on the innovation sequence $V_m$ to test the assumption on $u_2(\cdot)$, see below.

3. The state is estimated based on the GLR test result. Notice that the filter for state estimation is different from the mismatched filter used in step 1 for generating the sequence $V_m$, see below.

4. The algorithm is reinitialized upon detection and isolation of a change, see below.

### *2.2.1. The GLR test*

For the case of additive changes of unknown and unbounded magnitude embedded in gaussian noise, the log-likelihood ratio $l(k, k_s)$ is defined as (see Basseville & Nikiforov 1993, p. 241):

$$l(k, k_s) = \hat{\nu}(k, k_s) d(k, k_s) - \frac{1}{2} \hat{\nu}^2(k, k_s) J(k, k_s)$$

with

$$\begin{aligned} d(k, k_s) &= \Upsilon^T(k, k_s) \Sigma^{-1} V_m \\ J(k, k_s) &= \Upsilon^T(k, k_s) \Sigma^{-1} \Upsilon(k, k_s) \end{aligned}$$

where $k_s$ is the change point, $d(k, k_s)$ is the correlation between the innovation sequence $V_m$ and the signature sequence $\Upsilon(k, k_s)$; $\Sigma$ is the block matrix formed by the component covariance of the innovation sequence $V_m$; $J(k, k_s)$ is the Kullback-Leibler divergence; and $\hat{\nu}(k, k_s)$ is the magnitude estimate. Using the maximum likelihood estimate of the magnitude, the log-likelihood ratio can be rewritten as:

$$\begin{aligned} \sup_{\nu} \, l(k, k_s) &= \frac{1}{2} \frac{d^2(k, k_s)}{J(k, k_s)} \\ \hat{\nu}(k, k_s) &= \frac{d(k, k_s)}{J(k, k_s)} \end{aligned}$$

Assuming that the measurement noise and process noise are gaussian, the $l(k, k_s)$ have a non-central $\chi^2$ distribution with one degree of freedom and non-centrality parameter $\lambda = \nu^2 J(k, k_s)$.

Under the same assumption, the maximum likelihood estimate of the magnitude $\hat{\nu}$ has a gaussian distribution with covariance $\sigma^2 = \nu^{-2}J^{-1}(k, k_s)$ (see Basseville & Nikiforov 1993, p. 242):

$$\begin{aligned} \mathcal{L}(S(k, k_S)) &= \chi^2(1, \nu^2 J(k, k_s)) \\ \mathcal{L}(\hat{\nu}(k, k_s)) &= \mathcal{N}(\nu, \nu^{-2}J^{-1}(k, k_s)) \end{aligned}$$

Let $g(k)$ be the maximum log-likelihood ratio:

$$g(k) = \max_{k_s} \sup_{\nu} \; l(k, k_s)$$

Then, the hypothesis $H_1$ that a change occurred versus the hypothesis $H_0$ that no change occurred is tested using a sequential $\chi^2$ test:

$$\begin{aligned} g(k) > h(\alpha) &\quad : \quad \text{a change is detected and isolated} \\ g(k) < h(\alpha) &\quad : \quad \text{no change detected} \end{aligned}$$

The detection threshold $h(\alpha)$ is chosen based on the specified probability of false detection $\alpha$ and on the expected statistical distribution of the log-likelihood ratio if hypothesis $H_0$ is true. In the case of gaussian noise, this statistical distribution is a central chi-square distribution with one degree of freedom $\chi^2(1, 0)$.

### *2.2.2. State estimation*

The filter for state estimation is adjusted in function of the GLR test result. There's three cases to consider:

1. No change detected, i.e. the maximum log-likelihood ratio is less than the detection threshold. A dynamical change may (or may not) have occurred but if so, it's not detectable yet. Under theses circumstances, a filter with a high bandwidth is used.

2. A change is detected. The state estimate is corrected using the estimated dynamic profile of the change detected by the GLR test. After correction, the filter bandwidth is reduced for some period of time. This period of time depends on the lower bound of the time duration between switches and on the estimated change point.

3. A previously detected change appears to be a false alarm shortly after detection (i.e. the reinitialization of the detector). The correction of the state estimate is cancelled and the filter bandwidth returns to a high level.

The chosen filter for state estimation is a Kalman filter with a shaping filter. The shaping filter is an integrator of a random walk. The augmented state associated to the shaping filter is interpreted as the Kalman filter estimate of the unknown input $u_2$. The filter bandwidth is adjusted by setting appropriately the covariance of the random walk.

Upon detection of a change, the state estimate $x(k|k)$ is corrected using the information on the dynamic profile. Let's denote the uncorrected state estimate as $x(k|k)_\infty$. Then:

$$x(k|k) = x(k|k)_\infty + M(k,k_s)\hat{\Delta}U_2(k,k_s)$$

where

$$\begin{aligned} M(k,k_s) &\equiv (I - W(k)H)\Gamma(k,k_s) \\ \hat{\Delta}U_2(k,k_s) &= \hat{U}_{2\mathrm{GLR}} - \hat{U}_{2\mathrm{SF}} \end{aligned}$$

and

$$\begin{aligned} \Gamma(k,k_s) &\equiv \left[\begin{array}{cccc} G_2 & \Phi(k-1)G_2 & \cdots & \left(\prod_{j=0}^{k-k_s-1} \Phi(k-j-1)\right) G_2 \end{array}\right] \\ \Phi(k) &\equiv F(I - W(k)H) \end{aligned}$$

In the last equations, $\hat{U}_{2\mathrm{GLR}}$ is the dynamic profile estimated by the GLR detector, $\hat{U}_{2\mathrm{SF}}$ is the dynamic profile estimated by the shaping filter and $W(k)$ is the Kalman gain.

If the previously detected change appears to be a false alarm shortly after the detection, the state correction is cancelled. Let $x(k|k)_0$ be the corrected state, then the state without the correction is:

$$x(k|k) = x(k|k)_0 - (I - W(k)H)\Xi(k.k_0,k_s)\hat{\Delta}U_2(k_0,k_s)$$

with

$$\Xi(k,k_0,k_s) \equiv \left[\prod_{j=0}^{k-k_0-2} \Phi(k-j-1)\right] FM(k_0,k_s)$$

### *2.2.3. Signature in the innovation sequence*

Let's define $\bar{U}_2(k,k_s)$ as a dynamic profile with a normalized magnitude. The signature $\rho(k,k_s)$ in the innovation sequence is then defined as:

$$\rho(k,k_s) = H^\star\Gamma^\star(k,k_s)\bar{U}_2(k,k_s)$$

The corresponding sequence of signature $\Upsilon(k,k_s)$ can be written as:

$$\begin{aligned} \Upsilon(k,k_s) &= \Psi^\star(k,k_s)\bar{U}_2(k,k_s) \\ \Psi(k,k_s) &\equiv \left[\begin{array}{cccc} HG_2 & H\Phi(k-1)G_2 & \cdots & H\prod_{j=0}^{k-k_s-2}\Phi(k-j-1)G_2 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & HG_2 & H\Phi(k_s+1)G_2 \\ 0 & \cdots & 0 & HG_2 \end{array}\right] \end{aligned}$$

See the appendix for the derivation.

### *2.2.4. Reinitialization*

After detection and isolation of a change, the GLR test and the mismatched filter are reinitialized. A change is deemed "isolated" once the maximum likelihood ratio reaches some threshold. The latter threshold should be higher or equal to the detection threshold (see the GLR section).

A high detection threshold allows for a better estimation of the change magnitude and change point. Also, a high threshold delays the reinitialization after detection of a change, so a high threshold may allow the detection of a false alarm. However, in situation for which further changes are expected in the future, the reinitialization must occurred before the next change point. So, there's a trade-off in choosing the reinitialization threshold. This reinitialization requirement is restricting the application of the algorithm to situations where changes are occurring sparsely, i.e. the time duration between changes is bounded from below (Basseville & Nikiforov 1993, p. 57).

To reinitialize the GLR test, the likelihood ratios are reset to zero and only the signatures with a change point after the reinitialization are considered.

To reinitialize the mismatched filter, the dynamical model used by the mismatched filter is updated and the state estimate of the mismatched filter is corrected. The updated dynamical model uses the GLR estimated dynamic profile of $u_2(\cdot)$ instead of the assumption $u_2(\cdot) = 0$. The state estimate of the mismatched filter can be corrected in many ways as discussed by Caglayan & Lancraft 1983. The approach chosen here is a direct correction of the mismatched filter state estimate. This approach assumes that the maximum likelihood state estimates provided by the GLR detector can be trusted. Let $x^*(k|k)_\infty$ be the uncorrected mismatched filter state estimate. Then, the corrected state estimate is:

$$x^\star(k|k) = x^\star(k|k)_\infty + M^\star(k, k_s)\hat{U}_{2\mathrm{GLR}}$$

where

$$\hat{U}_{2\mathrm{GLR}} = \hat{\nu}(k, k_s)\bar{U}_2(k, k_s)$$

The matrix $M^\star(k, k_s)$ is defined as in the state estimation section (see above) but is computed using the matrices of the dynamical model used by the mismatched filter. If the dynamic profile is constant, the vector $\hat{U}_{2\mathrm{GLR}}$ can be replaced by a scalar $\hat{u}_2$ and the matrix $\Gamma^\star(k, k_s)$ involved in the computation of $M^\star(k, k_s)$ simplifies to:

$$\Gamma^\star(k, k_s) \equiv \left[\sum_{m=0}^{k-k_s} \left(\prod_{j=0}^{m-1} \Phi^\star(k-j-1)\right)\right] G_2^\star$$

### 2.2.5. *Reinitialization error*

In general, in a stochastic environment the estimated magnitude of the dynamic profile $\hat{U}_2$ and the true magnitude are not equal. The resulting error $\Delta U_2$ produces its own signature on the innovation sequence after the reinitialization (i.e. $k > k_0$). The error is expressed as:

$$\begin{aligned} \Delta U_2 &= U_2|_{\text{true}} - \hat{U}_{2\text{GLR}} \\ &= \bar{U}_2(k_0, k_s)\Delta\nu \end{aligned}$$

and its signature as:

$$\rho(k, k_s)_{\Delta U_2} = H^\star \left[\Xi^\star(k, k_0, k_s)\bar{U}_2(k_0, k_s) + \Gamma^\star(k, k_0+1)\bar{U}_2(k, k_0+1)\right] \Delta\nu$$

This signature can be added to the bank of signature used by the GLR detector to later detect and correct for the error $\Delta\nu$. The correction associated to this signature is expressed as:

$$x^\star(k|k) = x^\star(k|k)_\infty + \tilde{M}^\star \Delta\hat{\nu}$$

with

$$\tilde{M}^\star \equiv (I - W^\star(k)H) \left[\Xi^\star(k, k_0, k_s)\bar{U}_2(k_0, k_s) + \Gamma^\star(k, k_0)\bar{U}_2(k, k_0+1)\right]$$

### 2.2.6. *Recursive computation*

To alleviate the computational burden of the algorithm, the various equations can be rewritten in a recursive manner. Assuming the dynamic profile is constant and is equal to one ($u_2 = 1$):

$$\begin{aligned} \Gamma(k+1, k_s) &= G_2 + \Phi(k)\Gamma(k, k_s) \\ \Gamma(k_s+1, k_s) &= G_2 \end{aligned}$$

Normalized signature:

$$\rho(k, k_s) = H\Gamma(k, k_s)$$

Normalized signature due to a reinitialization error:

$$\rho(k, k_0)_{\Delta U_2} = H\left(\Xi(k, k_0, k_s) + \Gamma(k, k_0)\right)$$

with

$$\begin{aligned} \Xi(k+1, k_0, k_s) &= \Phi(k)\Xi(k, k_0, k_s) \\ \Xi(k_0+1, k_0, k_s) &= FM(k_0, k_s) \end{aligned}$$

Log-likelihood ratio:

$$\begin{aligned} l(k+1, k_s) &= \frac{1}{2}\frac{d^2(k+1, k_s)}{J(k+1, k_s)} \\ d(k+1, k_s) &= d(k, k_s) + \rho(k+1, k_s)S_{k+1}^{-1}v_m(k+1) \\ J(k+1, k_s) &= J(k, k_s) + \rho(k+1, k_s)S_{k+1}^{-1}\rho(k+1, k_s) \end{aligned}$$

where $v_m(k+1)$ is the $k+1$ innovation from the mismatched filter and $S_{k+1}$ is the corresponding covariance of the innovation.

### 2.3. Comments

A full implementation of a GLR test would imply a growing bank of signature. This is why for practical implementation the window limited GLR test is used. The window limited GLR test considers only the dynamic profiles with a switch time within some specified sliding window. The window limited GLR isn't a finite horizon technique. Because the likelihood ratios itself are computed against all the previous information, not just the information in the window (Basseville & Nikiforov 1993).

The approach chosen to reset the filter for state estimation after detection of a change is not optimal because only the information of the dynamic profile with the maximum likelihood estimate is used, however this simple approach as shown to give good results in some applications (Willsky 1986). An alternative to the current approach could be to build the pdf of the change using the likelihood ratio of each signature and then use this pdf to correct the state estimate (Caglayan & Lancraft 1983).

## 3. Simulation code

A simulation code is used to model a pursuit-evasion engagement between a pursuer and an evader. The model used in this work is restricted to a horizontal plane, the Earth is assumed to be flat (i.e. no Coriolis and centripete forces) and the gravitational and drag effects are neglected. The pursuer and the evader flight at constant speed, they are treated as point mass and their dynamic is assumed to be subject to a first order lag.

### 3.1. Dynamic

Let $a_P^c$ be the pursuer commanded acceleration, $a_P$ its achieved acceleration and $\tau_P$ be the time constant of the first order lag. The notation is similar for the evader but the evader quantities are denoted by the subscript $E$ instead of $P$. The non-linear equations describing the interception in Cartesian coordinates (denoted by the subscripts 1 and 2) are:

$$\dot{R}_{E1} = -V_P\cos(\gamma), \qquad \dot{R}_{E2} = V_P\sin(\gamma)$$
$$\dot{R}_{E1} = -V_E\cos(\beta), \qquad \dot{R}_{E2} = V_E\sin(\beta)$$

$$\begin{aligned} \dot{\gamma} &= \frac{a_P}{V_E} \\ \dot{\beta} &= \frac{a_E}{V_E} \\ \dot{a}_P &= \frac{a_P^c - a_P}{\tau_P} \\ \dot{a}_E &= \frac{a_E^c - a_E}{\tau_E} \end{aligned}$$

### 3.2. Estimator

Beside the GLR detector/estimator, a Kalman filter with a shaping filter (Kalman/SF) is implemented. The shaping filter is an integrator of a random walk. The matrices used by the Kalman/SF filter $(F, G, H)$ representing a linearized dynamic of the engagement are:

$$F(\Delta) = \mathcal{L}^{-1}(sI - A)^{-1}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{-1}{\tau_E} & 0 & \frac{1}{\tau_E} \\ 0 & 0 & 0 & \frac{-1}{\tau_P} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \int_0^{\Delta} F(\tau) B d\tau, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\tau_P} \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the state vector is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

The elements of the state vector are the relative lateral distance ($x_1$), the relative lateral velocity ($x_2$), the evader achieved acceleration ($x_3$), the missile achieved acceleration ($x_4$) and the evader commanded acceleration ($x_5$).

The measurements are subject to an additive gaussian noise. The gaussian noise is simulated by applying a Box-Muller transformation to numbers produced by a pseudo-random number generator having a uniform distribution. The angular measurement $\theta$ and its covariance $\sigma^2$ are converted in Cartesian $(y, \sigma_y^2)$ using the small angle approximation:

$$\begin{aligned} y &= R\theta \\ \sigma_y^2 &= (R\sigma^2)^2 \end{aligned}$$

### 3.3. Guidance laws

Two guidance laws are available and they are denoted DGL/1 and DGL/0. These guidance laws are derived from differential game theory and both require the assumption of perfect information to be optimal. These two guidance laws are expressed as:

$$a_P^c = a_P^{max} \mathrm{sign}(ZEM)$$

where ZEM stands for the Zero Effort Miss. The laws DGL/1 and DGL/0 differ in their way of computing the zero effort miss. The law DGL/1 includes the information on the evader acceleration in the computation of the ZEM (and assumes the acceleration will remain constant) while DGL/0 neglects the information on the evader acceleration. See Shinar & Shima 2000 and references their in for more details about these guidance laws.

## 4. Results

This section assesses the performance of the GLR detector/estimator against the Kalman/SF filter. The simulation parameters are chosen to represent a terminal guidance engagement between an anti-missile (pursuer) and a ballistic missile (evader). During the engagement, the evader attempts a single bang-bang maneuver; the switch time of this bang-bang maneuver is unknown to the pursuer.

### 4.1. Parameters

The parameters are the following:

| | | |
|---|---|---|
| initial range | : | $R = 20\ 000$ m |
| pursuer velocity | : | $v_P = 2300$ m/s |
| evader velocity | : | $v_E = 2700$ m/s |
| maximum pursuer acc. | : | $a_P^{max} = 30$ g |
| maximum evader acc. | : | $a_E^{max} = 15$ g |
| pursuer time constant lag | : | $\tau_P = 0.2$ s |
| evader time constant lag | : | $\tau_E = 0.2$ s |
| measurement frequency | : | $f = 100$ Hz |
| measurement noise covariance | : | $\sigma^2 = 0.1$ mrad |

The initial relative geometry is chosen to be a head-on engagement. The filters assume the zero initial conditions.

For the Kalman/SF filter, the process noise covariance matrix $Q_k$ is set to the covariance of the random walk of the shaping filter $\phi_n$. Following Zarchan 1997, $\phi_n$ is chosen to be $\phi_n = 4\frac{(a_E^{max})^2}{t_f}$; $t_f$ is the expected engagement duration.

$$Q_k = \int_0^\Delta F(\tau) Q F^T(\tau) d\tau, \qquad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_n \end{bmatrix}$$

where $\Delta$ is the time interval between measurements. The initial covariance matrix of the state estimate is set to:

$$P(0|0) = \begin{bmatrix} (R\sigma^2)^2 & 0 & 0 & 0 & 0 \\ 0 & (0.35 v_P)^2 & 0 & 0 & 0 \\ 0 & 0 & (a_E^{max})^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \phi_n \end{bmatrix}$$

For the GLR detector, the process noise covariance matrix of the mismatched filter $Q_k^\star$ is set to:

$$Q_k^\star = \int_0^\Delta F^\star(\tau) Q^\star F^{\star T}(\tau) d\tau, \quad Q^\star = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the initial covariance matrix of the state estimate is:

$$P^\star(0|0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### 4.2. Single shot kill probability

Contour maps of the Single Shot Kill (SSK) probability using in the homing loop the Kalman/SF filter and the guidance laws DGL/1 and DGL/0 are shown on Figure 1. The Figure 1 has been obtained by a Monte-Carlo simulation composed of 120 000 runs. The guidance law DGL/1 shows excellent performance except against evader maneuver switches happening during the last second of the engagement. After a switch, the estimate of the evader acceleration requires some time to converge to the new evader acceleration. During this period of time, the poor information on the evader acceleration leads the pursuer on a suboptimal trajectory. Against switches happening during the last second of the engagement, the actuator of the pursuer doesn't have enough time to correct the pursuer trajectory after the evader acceleration estimate converges to the new evader acceleration. The guidance law DGL/0 is more robust than DGL/1 with respect to the evader acceleration switch time since the information about the evader acceleration is not used by DGL/0. For most of the switch time however, the performance obtained DGL/0 are worse than the performance achieved by DGL/1.

As previously noted, the performance of DGL/1 law is very sensitive to the evader acceleration estimate. So, this guidance law is used to compare the performance of the GLR detector/estimator with the Kalman/SF filter performance.

On Figure 2, the performance achieved using these estimators is illustrated for the case of a switch happening 2 seconds before the end ($tgo_{sw} = 2$ s). In this situation, the measurement noise is the main contributor to the miss distance. After the switch is detected, the noise attenuation of the GLR detector/estimator gets better than the noise attenuation of the Kalman/SF filter. This explains the better performance obtained by using the GLR detector/estimator in this situation. It's interesting also to note that above a SSK probability of 0.95, the performance of the GLR estimator/detector drops abruptly. Upon investigation of this phenomenon, it has been found that the deterioration is due to a reinitialization error detected in a critical and narrow window of time. This critical window of detection happens between the moment when the evader initiate a switch

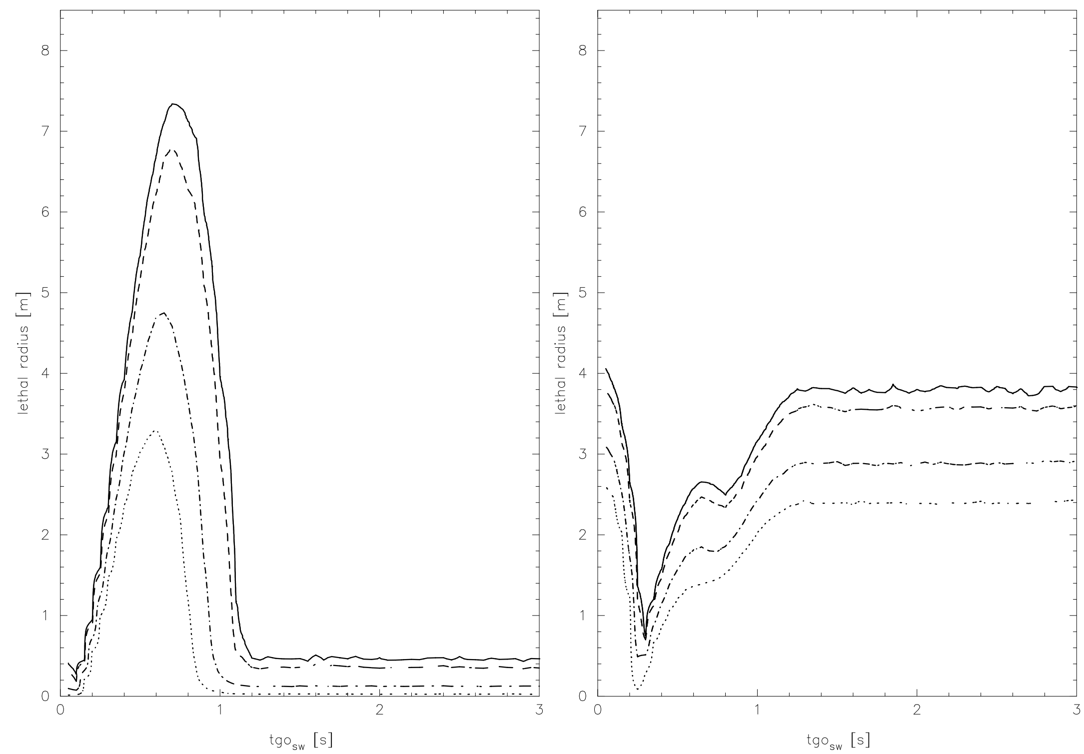


Fig. 1.— SSK probability versus the required pursuer lethal radius and the evader maneuver switch time using the Kalman/SF filter. On the left panel, the guidance law used is DGL/1, on the right panel the guidance law used is DGL/0. From top to bottom, the SSK probability of each curve is 0.95 (line), 0.90 (dash), 0.50 (dash-dot), and 0.10 (dot), respectively.

and the when moment the GLR could detect this switch. If the mismatched filter happens to be reinitialized during this window of time, the likelihood ratio of the GLR test gets also reinitialized. By reinitializing the likelihood ratio after the evader initiated a maneuver, the detector loses track of what the evader is doing; i.e. none of the GLR signatures are going to match the signature in the innovations coming from the mismatched filter after reinitialization. It happens that the detector is reinitialized in this critical window of time to correct a reinitialization error. This is the event responsible for the drop of performance above an SSK of 0.95. The solution to this problem is to not reset the likelihood ratios upon a correction due to a reinitialization error but rather to correct the likelihood ratios. Such a correction can be computed recursively but has not been implemented yet in the current GLR detector/estimator.

On Figure 3, the switch time is now set to a time-to-go of 0.5 seconds. In this case, the GLR detector/estimator is now worse than the Kalman/SF filter. Two reasons explain this behaviour. First of all, by the time the evader maneuver is detected and the filter bandwidth is reduced, the actuator command is already at maximum and there's not enough time left for the actuator to complete the correction of the pursuer trajectory. So, in such situation the performance of the GLR should be similar to the performance of the Kalman/SF filter. However, upon the detection of a change, the GLR detector not only reduces the bandwidth of the estimator, it also corrects the state estimate of the estimator. Some of these state corrections are poor and they are responsible for the deterioration of the GLR detector/estimator performance compares to the Kalman/SF filter performance.

### 4.3. Comments

A GLR detector/estimator allows using effectively the information that upon initiation of a maneuver by the evader during the last few seconds of the engagement, the evader is unlikely to initiate a different maneuver afterward. Doing so would prevent the evader of significantly changing it's trajectory and it would then be less likely to avoid interception.

The mismatched filter used by the GLR detector/estimator must be design with care, because it has been observed (not shown in this work) that the GLR detector/estimator performance are highly dependent on the mismatched filter. This is to be expected since the GLR detector is gathering all its information from the innovations generated by this single mismatched filter.

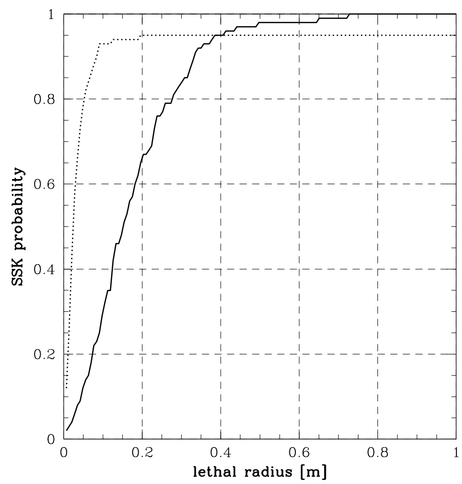


Fig. 2.— SSK probability distribution versus the required pursuer lethal radius for a evader maneuver switch time hapenning 2.0 secondes before the end. The distribution obtained using the Kalman/SF filter line) and the GLR detector/estimator (dot) are shown. This figure has been obtained by a Monte-Carlo simulation of 100 runs

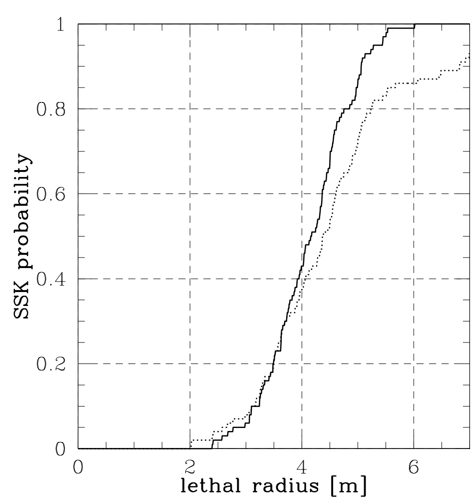


Fig. 3.— SSK probability distribution versus the required pursuer lethal radius for a evader maneuver switch time happening 0.5 secondes before the end. The distribution obtained using the Kalman/SF filter (line) and the GLR detector/estimator (dot) are shown. The figure has been obtained by a Monte-Carlo simulation of 100 runs.

## 5. Conclusion

The use of a GLR detector/estimator in the homing loop allows to improve the homing accuracy provided that the evader final maneuver starts at the sufficiently large time before the end time. Otherwise, the achieved performances using the GLR detector/estimator are similar or worse to the Kalman/SF filter performance. Unfortunately, the evader final maneuvers heavily affecting the pursuer performance (SSK probability) have an onset time close to the end time. So, the use of a GLR detector/estimator in the homing loop doesn't lead to an improvement of the pursuer performance over the Kalman/SF filter against the worse evader maneuvers.

A way to improve the homing performance against the worse evader maneuvers could be to use an estimator with a shorter delay of detection. However, a minimum delay of detection cannot be avoided. An idea of the minimum delay of detection can be obtained by measuring the amount of information available about the evader maneuver. The Kullback-Leibler divergence can be used as a weak measure of this information. Looking at the behaviour of the Kullback-Leibler divergence versus the detection delay of the GLR test (not shown in this work), it appears there's not much more room left to reliably improve the delay of detection. Moreover, it has been shown that the delay of detection of the GLR test asymptotically reaches the optimal lower bound on detection delay subject to the false alarm constraint as the false alarm constraint goes to zero (Lai & Shan 1999).

A different possibility to improve the overall performance of the homing loop would be to match a robust GLR detector/estimator with an adaptive guidance law. Such an adaptive guidance law would be cautious ("high bandwidth") about its assumptions on the evader state before the detection of a maneuver; after the detection, the guidance law would use more restrictive assumptions ("low bandwidth"). This scheme effectively uses the assumption that the interval of time between evader maneuver switches is bounded from below for both the estimator and the guidance law. Such an adaptive guidance law could be a combination of the DGL/0 and DGL/1 law. Before the switch detection, DGL/0 is used; after the detection, DGL/1 is used. Another more elegant possibility would be to use the delayed information guidance law proposed by Shinar & Shima 2000. This guidance law includes a time constant representing the estimation error as a time delay between the true and the estimated evader acceleration. This time constant could be chosen to be large before detection of a change (the delayed guidance law will then tend toward DGL/0) and be reduced after detection of a change. Work is currently being pursued into testing this approach.

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## Appendix

Let's define a Kalman filter assuming that $u_2(\cdot) = 0$. An asterisk denotes the equations of this filter. Then, the state prediction and the corresponding innovation are expressed as:

$$\begin{aligned} \hat{x}^*(k+1|k) &= F[I - W(k)H]\hat{x}^*(k|k-1) + FW(k)y(k) + G_1u_1(k) \\ v^*(k+1) &= y(k+1) - H\hat{x}^*(k+1|k) \end{aligned}$$

where $W(k)$ is the Kalman gain. For ease of notation, let's define

$$\Phi(k) = F[I - W(k)H]$$

Similarly, we can defined a second filter with perfect information on $u_2$:

$$\begin{aligned} \hat{x}(k+1|k) &= \Phi(k)\hat{x}(k|k-1) + FW(k)y(k) + G_1u_1(k) + G_2u_2(k) \\ v(k+1) &= y(k+1) - H\hat{x}(k+1|k) \end{aligned}$$

By combining the equations from the two filters, we can write:

$$\begin{aligned} v^*(k+1) &= v(k+1) + H\left[\hat{x}(k+1|k) - \hat{x}^*(k+1|k)\right] \\ &= v(k+1) + H\left[\Phi(k)\hat{x}(k|k-1) - \Phi(k)\hat{x}^*(k|k-1) + G_2u_2(k)\right] \end{aligned}$$

Let express the last equation as a function of an arbitrary time $k_s$ in the past:

$$\begin{aligned} v^*(k+1) &= v(k+1) + H\left[\left(\prod_{j=0}^{k-k_s-1}\Phi(k-j)\right)\Delta(k_s+1|k_s) + \sum_{m=k_s}^{k}\left(\prod_{j=0}^{k-m-1}\Phi(k-j)\right)G_2u_2(m)\right] \\ \Delta(k_s+1|k_s) &= \hat{x}(k_s+1|k_s) - \hat{x}^*(k_s+1|k_s) \end{aligned}$$

Suppose $\Delta(k_s+1|k_s) = 0$ (i.e. no initialization error) to obtain:

$$v^*(k+1) = v(k+1) + H\left[\sum_{m=k_s}^{k}\left(\prod_{j=0}^{k-m-1}\Phi(k-j)\right)G_2u_2(m)\right]$$

The last equation can be rewritten in matrix form as:

$$V^*(k,k_s) = V(k,k_s) + \Psi(k,k_s)U_2(k,k_s)$$

with

$$V^*(k,k_s) \equiv \begin{bmatrix} v^*(k) \\ \vdots \\ v^*(k_s+1) \end{bmatrix}, \quad V(k,k_s) \equiv \begin{bmatrix} v(k) \\ \vdots \\ v(k_s+1) \end{bmatrix}, \quad U_2(k,k_s) \equiv \begin{bmatrix} u_2(k-1) \\ \vdots \\ u_2(k_s) \end{bmatrix}$$

$$\Psi(k,k_s) \equiv \begin{bmatrix} HG_2 & H\Phi(k-1)G_2 & \cdots & H\prod_{j=0}^{k-k_s-2}\Phi(k-j-1)G_2 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & HG_2 & H\Phi(k_s+1)G_2 \\ 0 & \cdots & 0 & HG_2 \end{bmatrix}$$

In the last equation, the sequence of innovation $V$ has a gaussian distribution with zero mean and $U_2$ is a dynamic profile. Consequently, the sequence of signature $\Upsilon$ in the innovation sequence $V^*$ is:

$$\Upsilon(k,k_s) = \Psi(k,k_s)U_2(k,k_s)$$